Sphere Packing in Lean

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Contents

Contents i

1 Overview 1

2 Sphere Packings 2

3 Density of Packings 8

4 The E8 Lattice 11

5 Fourier Analysis 14

6 Cohn-Elkies Bounds 17

7 Modular Forms 19

8 Fourier Eigenfunctions with Double Zeroes 36

9 Proof of the Optimal Function Inequalities 52

Dependency Graph 59

Blueprint Summary 60

Blueprint Bibliography 61

1 Overview

This document ports the original Sphere Packing in Lean TeX blueprint to Verso Blueprints and tracks the same mathematical narrative.

The project formalizes Maryna Viazovska's theorem that no packing of unit balls in $\mathbb{R}^8$ has density greater than the E_8 lattice packing; see Viazovska (2017). The global upper bound comes from the linear programming method of Cohn and Elkies (2003), and the crucial input is an explicit optimal auxiliary function built from modular forms.

Compared with the upstream TeX source, this Verso copy reorganizes the blueprint into linked nodes and groups so that the dependency graph and summary are rendered directly from the structured document.

2 Sphere Packings

The sphere packing problem is a classic optimisation problem with widespread applications that go well beyond mathematics. The task is to determine the densest possible arrangement of spheres in a given space. It remains unsolved in all but finitely many dimensions.

It was famously determined, in Viazovska (2017), that the optimal arrangement in $\mathbb{R}^8$ is given by the E_8 lattice. The result is strongly dependent on the Cohn-Elkies linear programming bound, Theorem 3.1 in Cohn and Elkies (2003), which, if a $\mathbb{R}^d \to \mathbb{R}$ function satisfying certain conditions exists, bounds the optimal density of sphere packings in $\mathbb{R}^d$ in terms of it. The proof in Viazovska (2017) uses the theory of modular forms to construct a function that can be used to bound the density of all sphere packings in $\mathbb{R}^8$ above by the density of the E_8 lattice packing. This then allows us to conclude that no packing in $\mathbb{R}^8$ can be denser than the E_8 lattice packing.

This subsection gives an overview of the setup of the problem, both informally and in Lean. Throughout this blueprint, $\mathbb{R}^d$ denotes the Euclidean vector space equipped with distance $\|\cdot\|$ and Lebesgue measure $\mathrm{Vol}(\cdot)$. For any $x \in \mathbb{R}^d$ and $r \in \mathbb{R}_{>0}$, we denote by $B_d(x, r)$ the open ball in $\mathbb{R}^d$ with center x and radius r. While we will give a more formal definition of a sphere packing below, the underlying idea is that it is a union of balls of equal radius centered at points that are far enough from each other that the balls do not overlap.

Arguably the most important definition in this subsection is that of packing density, which measures which portion of d-dimensional Euclidean space is covered by a given sphere packing. Taking the supremum over all packings gives the sphere packing constant, which is the quantity we are interested in optimising.

Definition 2.1

Given a set $X \subset \mathbb{R}^d$ and a real number $r > 0$ such that $\|x - y\| \geq r$ for all distinct $x, y \in X$, we define the sphere packing $\mathcal{P}(X)$ with centres at X to be the union of all open balls of radius r centred at points in X:

$$\mathcal{P}(X) := \bigcup_{x \in X} B_d(x, r)$$

We now define a notion of density for bounded regions of space by considering the density inside balls of finite radius.

Definition 2.2

The finite density of a packing $\mathcal{P}$ is defined as

$$\Delta_{\mathcal{P}}(R) := \frac{\text{Vol}(\mathcal{P} \cap B_d(0,R))}{\text{Vol}(B_d(0,R))}, \quad R > 0.$$

As intuitive as it seems to take the density of a packing to be the limit of the finite densities as the radius of the ball goes to infinity, it is not immediately clear that this limit exists. Therefore, we define the density of a sphere packing as a limit superior instead.

Definition 2.3

We define the density of a packing $\mathcal{P}$ as the limit superior

$$\Delta_{\mathcal{P}} := \limsup_{R\to\infty} \Delta_{\mathcal{P}}(R).$$

We may now define the sphere packing constant, the quantity that the sphere packing problem requires us to compute.

Definition 2.4

The sphere packing constant is defined as the supremum of packing densities over all possible packings:

$$\Delta_d := \sup_{\substack{\mathcal{P}\subset\mathbb{R}^d \\ \text{sphere packing}}} \Delta_{\mathcal{P}}.$$

Definition 2.5

Given a sphere packing $\mathcal{P}(X)$ with separation radius r, we define the scaled packing with respect to a real number $c > 0$ to be the packing $\mathcal{P}(cX)$, where $cX = \{cx \in V \mid x \in X\}$ has separation radius cr.

Lemma 2.6

Let $\mathcal{P}(X)$ be a sphere packing and c a positive real number. Then, for all $R > 0$,

$$\Delta_{\mathcal{P}(cX)}(cR) = \Delta_{\mathcal{P}(X)}(R).$$

Proof for Lemma 2.6

The proof follows by direct computation:

$$\Delta_{\mathcal{P}(cX)}(cR) = \frac{\text{Vol}(\mathcal{P}(cX) \cap B_d(0, cR))}{\text{Vol}(B_d(0, cR))} = \frac{c^d \cdot \text{Vol}(\mathcal{P}(X) \cap B_d(0, R))}{c^d \cdot \text{Vol}(B_d(0, R))} = \Delta_{\mathcal{P}(X)}(R).$$

The second equality follows from the fact that scaling a measurable set by a factor of c scales its volume by a factor of c^d, together with $\mathcal{P}(cX) \cap B_d(0, cR) = c \cdot (\mathcal{P}(X) \cap B_d(0, R))$.

Lemma 2.7

Let $\mathcal{P}(X)$ be a sphere packing and c a positive real number. Then the density of the scaled packing $\mathcal{P}(cX)$ is equal to the density of the original packing $\mathcal{P}(X)$.

Proof for Lemma 2.7

One can show, using relatively unsophisticated real analysis, that

$$\limsup_{R\to\infty} \Delta_{\mathcal{P}(cX)}(R) = \limsup_{cR\to\infty} \Delta_{\mathcal{P}(cX)}(cR).$$

Lemma Lemma 2.6 tells us that $\Delta_{\mathcal{P}(cX)}(cR) = \Delta_{\mathcal{P}(X)}(R)$ for every $R > 0$. Therefore,

$$\limsup_{cR\to\infty} \Delta_{\mathcal{P}(cX)}(cR) = \limsup_{cR\to\infty} \Delta_{\mathcal{P}(X)}(R) = \limsup_{R\to\infty} \Delta_{\mathcal{P}(X)}(R),$$

where the second equality is the result of a similar change of variables.

Therefore, as expected, we do not need to worry about the separation radius when constructing sphere packings. This will be useful when we attempt to construct the optimal sphere packing in $\mathbb{R}^8$, and even more so when attempting to formalise this construction, because the underlying structure of the packing is given by the E_8 lattice, which has separation radius $\sqrt{2}$.

Lemma 2.8

$$\Delta_d = \sup_{\substack{\mathcal{P}\subset\mathbb{R}^d \\ \text{sphere packing} \\ \text{sep. rad.}=1}} \Delta_{\mathcal{P}}$$

Proof for Lemma 2.8

That the supremum over packings of unit density is at most the sphere packing constant is obvious. For the reverse inequality, let $\mathcal{P}(X)$ be any sphere packing with separation radius r. By Lemma 2.7, the density of $\mathcal{P}(X)$ is equal to that of the scaled packing $\mathcal{P}\left(\frac{X}{r}\right)$. Since the scaled packing has separation radius 1, its density is at most the supremum over all packings of unit density, and therefore the same is true of $\mathcal{P}(X)$.

We begin by defining what a lattice is in Euclidean space.

Definition 2.9

We say that an additive subgroup $\Lambda \leq \mathbb{R}^d$ is a lattice if it is discrete and its $\mathbb{R}$-span contains all the elements of $\mathbb{R}^d$.

There is also a corresponding dual notion, which becomes relevant when we start doing Fourier analysis on functions over lattices.

Definition 2.10

The dual lattice of a lattice Λ is the set

$$\Lambda^* := \left\{ v \in \mathbb{R}^d \mid \forall l \in \Lambda, \langle v, l \rangle \in \mathbb{Z} \right\}.$$

We can now define periodic sphere packings.

Definition 2.11

We say that a sphere packing $\mathcal{P}(X)$ is Λ-periodic if there exists a lattice $\Lambda \subset \mathbb{R}^d$ such that for all $x \in X$ and $y \in \Lambda$, we have $x+y \in X$.

There is a natural definition of density for periodic sphere packings, namely the local density of balls in a fundamental domain. However, a priori the density of sphere packing above need not to coincide with this alternative definition. In Theorem 3.5, we will prove that this is the case.

Now that we have simplified the process of computing the packing densities of specific packings, we can simplify the computation of the sphere packing constant. It turns out that once again, periodicity is key.

Definition 2.12

The periodic sphere packing constant is defined to be

$$\Delta_d^{\text{periodic}} := \sup_{\substack{P \subset \mathbb{R}^d \\ \text{periodic packing}}} \Delta_P.$$

Theorem 2.13

For all d, the periodic sphere packing constant in $\mathbb{R}^d$ is equal to the sphere packing constant in $\mathbb{R}^d$.

Proof for Theorem 2.13

The following proof was written by Junyan Xu in the thread for Issue #80[1] of the repository.Given an arbitrary packing P of upper density Δ and an arbitrary $\epsilon > 0$, we want to produce a periodic

[1] https://github.com/thefundamentaltheor3m/Sphere-Packing-Lean/issues/80

packing with density at least $\Delta - \epsilon$. Given a set S (of nonzero volume), we shall call the quantity $\frac{|S \cap P|}{|S|}$ the density of P in S, where $|S|$ denotes the volume of S.Choose $\ell > 0$ such that

$$\frac{(\ell - 4r)^d}{\ell^d} > 1 - \frac{\epsilon}{3}$$

(where r is the radius of the balls in P and d is the dimension). Then choose $R_0 > 0$ such that for all $R > R_0$, we have

$$\frac{\left(R + \sqrt{d}\,\ell\right)^d - \left(R - \sqrt{d}\,\ell\right)^d}{R^d} < \frac{\epsilon}{3}$$

By definition of upper density, there exists $R > R_0$ such that the density of P in $B(0, R)$ is at least $\Delta - \frac{\epsilon}{3}$.The Euclidean space is covered by disjoint hypercubes of side length ℓ of the form $\prod_{i=1}^{d} [n_i \ell, (n_i + 1)\ell)$ with $n_i \in \mathbb{Z}$. Since these hypercubes have diameter $\sqrt{d}\ell$, those that intersect $B(0, R)$ must be contained in $B(0, R + \sqrt{d}\ell)$. Thus the hypercubes contained in $B(0, R + \sqrt{d}\ell)$ cover $B(0, R)$. Similarly, the hypercubes not contained in $B(0, R)$ cannot intersect $B(0, R - \sqrt{d}\ell)$. Denote the union of hypercubes contained in $B(0, R + \sqrt{d}\ell)$ by T and the union of those contained in $B(0, R)$ by S, we then have

$$S \subseteq B(0, R) \subseteq T \subseteq B(0, R + \sqrt{d}\ell)$$

and

$$T \setminus S \subseteq B(0, R + \sqrt{d}\ell) \setminus B(0, R - \sqrt{d}\ell)$$

The density of P in S is

$$\begin{aligned} \frac{|S \cap P|}{|S|} &\geq \frac{|B(0, R) \cap P| - |B(0, R) \setminus S|}{|B(0, R)|} \\ &> \left(\Delta - \frac{\epsilon}{3}\right) - \frac{|T \setminus S|}{|B(0, R)|} \\ &> \left(\Delta - \frac{\epsilon}{3}\right) - \frac{\epsilon}{3} \end{aligned}$$

Therefore, the density of P in some hypercube $C \subseteq S$ is at least $\Delta - \frac{2\epsilon}{3}$. If a ball in P (thus of diameter $2r$) is not completely contained in C, it cannot intersect the hypercube C' with the same center as C with side length $\ell - 4r$, so excluding these balls from $C \cap P$ decreases the density in C by at most $\frac{|C \setminus C'|}{|C|} = \frac{\ell^d - (\ell - 4r)^d}{\ell^d} < \frac{\epsilon}{3}$, and gives rise to a periodic packing of density $> \Delta - \epsilon$ with C as a fundamental domain.

Thus, one can show a sphere packing to be optimal by showing its density to be equal to the periodic sphere packing constant instead of the regular

sphere packing constant. Determining the periodic constant is easier than determining the general constant, as we shall see when investigating the linear programming bounds derived by Cohn and Elkies in Cohn and Elkies (2003).

With the terminologies above, we can state the main theorem of this project.

Theorem 2.14

All periodic packing $\mathcal{P} \subseteq \mathbb{R}^8$ has density satisfying $\Delta_{\mathcal{P}} \leq \Delta_{E_8} = \frac{\pi^4}{384}$, the density of the E_8 sphere packing; see Definition 4.9.

Proof for Theorem 2.14

Directly follows from Theorem 6.2 applied to the function $f(x) = g(x/\sqrt{2})$ of Theorem 6.3.

Corollary 2.15

All packing $\mathcal{P} \subseteq \mathbb{R}^8$ has density satisfying $\Delta_{\mathcal{P}} \leq \Delta_{E_8}$.

Proof for Corollary 2.15

This is a direct consequence of Theorem 2.13 and Theorem 2.14.

Corollary 2.16

$\Delta_8 = \Delta_{E_8}$.

Proof for Corollary 2.16

By definition, $\Delta_{E_8} \leq \Delta_8$, while the upper-bound-E8 corollary Corollary 2.15 shows that $\Delta_8 = \sup_{\text{packing } \mathcal{P}} \leq \Delta_{E_8}$, and the result follows.

3 Density of Packings

The definition of density for sphere packings is inconvenient to work with, especially when our packing is a structured one, such as a periodic or lattice packing. This section fixes this problem.

Note that some of the proofs in this section have only been sketched. The finer details are formalised in Lean.

Observe that the finite density evaluated at some $R > 0$ measures the density of sphere packings within a bounded open ball of radius R. As one might expect, there is a relationship between the finite density and the number of centers in the ball of radius R.

Lemma 3.1

For any $R > 0$,

$$\left|X \cap \mathcal{B}_d\left(R - \frac{r}{2}\right)\right| \cdot \frac{\mathrm{Vol}\left(\mathcal{B}_d\left(\frac{r}{2}\right)\right)}{\mathrm{Vol}(\mathcal{B}_d(R))} \leq \Delta_{\mathcal{P}}(R) \leq \left|X \cap \mathcal{B}_d\left(R + \frac{r}{2}\right)\right| \cdot \frac{\mathrm{Vol}\left(\mathcal{B}_d\left(\frac{r}{2}\right)\right)}{\mathrm{Vol}(\mathcal{B}_d(R))}.$$

This is the basic estimate for Definition 2.2.

Proof for Lemma 3.1

The high-level idea is to prove that $\mathcal{P} \cap \mathcal{B}_d(R) = \left(\bigcup_{x \in X} \mathcal{B}_d\left(x, \frac{r}{2}\right)\right) \subseteq \bigcup_{x \in X \cap \mathcal{B}_d\left(R + \frac{r}{2}\right)} \mathcal{B}_d\left(x, \frac{r}{2}\right)$, and a similar inequality for the lower bound. The rest follows by rearranging and using the fact that the balls are pairwise disjoint in SpherePacking.

Suppose further that X is a periodic packing with respect to the lattice $\Lambda \subseteq \mathbb{R}^d$. Let $\mathcal{D}$ be a bounded fundamental region of Λ, say the parallelopiped $[0, 1]^n \Lambda$, and let L be a bound on the norm of vectors in $\mathcal{D}$, that is, a number satisfying $\forall x \in \mathcal{D}, \|x\| \leq L$.

Lemma 3.2

For all R, we have the following inequality relating the number of lattice points from Λ in a ball with the volume of the ball and the fundamental region $\mathcal{D}$:

$$\frac{\mathrm{Vol}(\mathcal{B}_d(R - L))}{\mathrm{Vol}(\mathcal{D})} \leq |\Lambda \cap \mathcal{B}_d(R)| \leq \frac{\mathrm{Vol}(\mathcal{B}_d(R + L))}{\mathrm{Vol}(\mathcal{D})}.$$

Proof for Lemma 3.2

For the first inequality, it suffices to prove that $\mathcal{B}_d(R-L) \subseteq \bigcup_{x\in\Lambda\cap\mathcal{B}_d(R)}(x+\mathcal{D})$, since the cosets on the right are disjoint. For a vector $v \in \mathcal{B}_d(R-L)$, we have $\|v\| < R-L$ by definition. Since $\mathcal{D}$ is a fundamental domain, there exists a lattice point $x \in \Lambda$ such that $v \in x+\mathcal{D}$. Rearranging gives $v-x \in \mathcal{D}$, which means $\|v-x\| \le L$. By the triangle inequality, $\|x\| < R$, so $x \in \mathcal{B}_d(R)$. /- source paragraph break -/ For the second inequality, we prove that $\bigcup_{x\in\Lambda\cap\mathcal{B}_d(R)}(x+\mathcal{D}) \subseteq \mathcal{B}_d(R+L)$. Consider a vector $x \in \Lambda\cap\mathcal{B}_d(R)$ and a vector $y \in x+\mathcal{D}$. Then $\|x\| < R$ and $\|y-x\| \le L$, so $\|y\| < R+L$, concluding the proof.

To link Lemma 3.1 and Lemma 3.2, we need a lemma relating $|\Lambda\cap\mathcal{B}|$ with $|X\cap\mathcal{B}|$, which is what the following lemma does.

Lemma 3.3

For all R, we have the following inequality relating the number of points from X periodic with respect to Λ in a ball with the number of points from Λ:

$$|\Lambda\cap\mathcal{B}_d(R-L)|\,|X/\Lambda| \le |X\cap\mathcal{B}_d(R)| \le |\Lambda\cap\mathcal{B}_d(R+L)|\,|X/\Lambda|\,.$$

Proof for Lemma 3.3

For the first inequality, we notice that $\bigcup_{x\in\Lambda\cap\mathcal{B}_d(R-L)}(x+\mathcal{D}) \subseteq \mathcal{B}_d(R)$, because for $x \in \Lambda\cap\mathcal{B}_d(R-L)$ and $y \in x+\mathcal{D}$, we have $\|x\| < R-L$ and $\|y-x\| \le L$, so $\|y\| < R$ by the triangle inequality. Intersecting both sides with X and simplifying, we get

$$\left(\bigcup_{x\in\Lambda\cap\mathcal{B}_d(R-L)}(x+\mathcal{D})\right)\cap X = \bigcup_{x\in\Lambda\cap\mathcal{B}_d(R-L)}((x+\mathcal{D})\cap X) \subseteq \mathcal{B}_d(R)\cap X.$$

Taking cardinalities and noting that $|(x+\mathcal{D})\cap X| = |X/\Lambda|$ for all x, we obtain $|\Lambda\cap\mathcal{B}_d(R-L)||X/\Lambda| \le |X\cap\mathcal{B}_d(R)|$. /- source paragraph break -/ The proof of the second inequality is similar. We again observe that $\mathcal{B}_d(R) \subseteq \bigcup_{x\in\Lambda\cap\mathcal{B}_d(R+L)}(x+\mathcal{D})$, which follows from the tiling property of a fundamental domain. Intersecting both sides with X and taking cardinalities concludes the proof. /- source paragraph break -/ There are several technicalities when formalising in Lean, such as having to prove that $|\Lambda\cap\mathcal{B}_d(R)|$ is countable and finite. Those are handled in `aux3`.

When we combine the inequalities above, we need one additional computational lemma.

Lemma 3.4

For any constant $C > 0$, we have

$$\lim_{R\to\infty} \frac{\text{Vol}(\mathcal{B}_d(R))}{\text{Vol}(\mathcal{B}_d(R+C))} = 1.$$

Proof for Lemma 3.4

Write out the formula for the volume of a ball and simplify. More specifically, we have $\text{Vol}(\mathcal{B}_d(R)) = R^d\pi^{d/2}/\Gamma\left(\frac{d}{2}+1\right)$, so $\text{Vol}(\mathcal{B}_d(R))/\text{Vol}(\mathcal{B}_d(R+C)) = R^d/(R+C)^d = \left(1-\frac{1}{R+C}\right)^d$, which tends to 1.

Finally, we can relate the density of a periodic sphere packing to the natural definition of density given by any of its fundamental domains.

Theorem 3.5

For a periodic sphere packing $\mathcal{P} = \mathcal{P}(X)$ with centers X periodic with respect to the lattice Λ and separation r,

$$\Delta_{\mathcal{P}} = |X/\Lambda| \cdot \frac{\text{Vol}(\mathcal{B}_d(r/2))}{\text{Vol}(\mathbb{R}^d/\Lambda)}.$$

This identifies Definition 2.3 for periodic packings.

Proof for Theorem 3.5

Fix any fundamental domain $\mathcal{D}$ (induced by any basis) of the lattice Λ. Combining Lemma 3.1, Lemma 3.2 and Lemma 3.3, we get the following inequality for the finite density:

$$|X/\Lambda|\cdot\frac{\text{Vol}(\mathcal{B}_d(r/2))}{\text{Vol}(\mathbb{R}^d/\Lambda)}\cdot\frac{\text{Vol}(\mathcal{B}_d(R-r/2-2L))}{\text{Vol}(\mathcal{B}_d(R))} \le \Delta_{\mathcal{P}}(R) \le |X/\Lambda|\cdot\frac{\text{Vol}(\mathcal{B}_d(r/2))}{\text{Vol}(\mathbb{R}^d/\Lambda)}\cdot\frac{\text{Vol}(\mathcal{B}_d(R+r/2+2L))}{\text{Vol}(\mathcal{B}_d(R))}.$$

Taking limit on both sides as $R \to \infty$ and apply the Sandwich theorem and Lemma 3.4, we get

$$\Delta_{\mathcal{P}} = \limsup_{R\to\infty} \Delta_{\mathcal{P}}(R) = \lim_{R\to\infty} \Delta_{\mathcal{P}}(R) = |X/\Lambda| \cdot \frac{\text{Vol}(\mathcal{B}_d(r/2))}{\text{Vol}(\mathbb{R}^d/\Lambda)}.$$

4 The E8 Lattice

There are several equivalent definitions of the E_8 lattice. Below, we formalise two of them and prove that they are equivalent.

Definition 4.1

We define the E_8 lattice as a subset of $\mathbb{R}^8$ by

$$\Lambda_8 = \{(x_i) \in \mathbb{Z}^8 \cup (\mathbb{Z} + \tfrac{1}{2})^8 | \ \sum_{i=1}^{8} x_i \equiv 0 \pmod 2\}.$$

Definition 4.2

We define the E_8 basis vectors to be the set of vectors

$$\mathcal{B}_8 = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

Theorem 4.3

The two definitions above coincide, i.e. $\Lambda_8 = \text{span}_{\mathbb{Z}}(\mathcal{B}_8)$.

Proof for Theorem 4.3

We prove that each side contains the other. /- source paragraph break -/ For a vector $\vec{v} \in \Lambda_8 \subseteq \mathbb{R}^8$, we have $\sum_i \vec{v}_i \equiv 0 \pmod 2$ and all coordinates are either integers or half-integers. After some modulo arithmetic, it can be seen that $\mathcal{B}_8^{-1}\vec{v}$ has integer coordinates, so $\vec{v} \in \text{span}_{\mathbb{Z}}(\mathcal{B}_8)$. /- source paragraph break -/ For the opposite direction, write $\vec{v} = \sum_i c_i \mathcal{B}_8^i \in \text{span}_{\mathbb{Z}}(\mathcal{B}_8)$ with the c_i integers and $\mathcal{B}_8^i$ the i-th basis vector. Expanding the definition gives

$\vec{v} = \left(c_1 - \frac{1}{2}c_7, -c_1 + c_2 - \frac{1}{2}c_7, \cdots, -\frac{1}{2}c_7\right)$. Again, after some modulo arithmetic, it can be seen that $\sum_i \vec{v}_i$ is 0 modulo 2 and that the coordinates are all either integers or all half-integers. /- source paragraph break -/ This proof does not depend on $\mathcal{B}_8$ being linearly independent.

In this section, we establish basic properties of the E_8 lattice and the $\mathcal{B}_8$ vectors.

Lemma 4.4

$\mathcal{B}_8$ is an $\mathbb{R}$-basis of $\mathbb{R}^8$. This uses Definition 4.2.

Proof for Lemma 4.4

It suffices to prove that $\mathcal{B}_8 \in \mathrm{GL}_8(\mathbb{R})$. We do this by explicitly defining the inverse matrix $\mathcal{B}_8^{-1}$ and proving $\mathcal{B}_8\mathcal{B}_8^{-1} = I_8$, which implies that $\det(\mathcal{B}_8)$ is nonzero. See the Lean code for more details.

Lemma 4.5

Λ_8 is an additive subgroup of $\mathbb{R}^8$.

Proof for Lemma 4.5

This follows trivially from the fact that $\Lambda_8 \subseteq \mathbb{R}^8$ is the $\mathbb{Z}$-span of $\mathcal{B}_8$ and hence an additive group.

Lemma 4.6

All vectors in Λ_8 have norm of the form $\sqrt{2n}$, where n is a nonnegative integer. This uses Theorem 4.3.

Proof for Lemma 4.6

Writing $\vec{v} = \sum_i c_i \mathcal{B}_8^i$, we have $\|v\|^2 = \sum_i \sum_j c_i c_j (\mathcal{B}_8^i \cdot \mathcal{B}_8^j)$. Computing all 64 pairs of dot products and simplifying gives a large quadratic form in the c_i with even integer coefficients, concluding the proof.

Lemma 4.7

$c\Lambda_8$ is discrete, that is, the subspace topology induced by its inclusion into $\mathbb{R}^8$ is the discrete topology.

Proof for Lemma 4.7

Since Λ_8 is a topological group and $+$ is continuous, it suffices to prove that $\{0\}$ is open in Λ_8. This follows from the existence of an open ball $\mathcal{B}(\sqrt{2}) \subseteq \mathbb{R}^8$ around zero containing no other lattice points, since the shortest nonzero vector has norm $\sqrt{2}$.

Lemma 4.8

$c\Lambda_8$ is a $\mathbb{Z}$-lattice, that is, it is discrete and spans $\mathbb{R}^8$ over $\mathbb{R}$.

Proof for Lemma 4.8

The first part is by Lemma 4.7, and the second part follows from the fact that $\mathcal{B}_8$ is a basis (Lemma 4.4) and hence linearly independent over $\mathbb{R}$.

Prove E_8 is unimodular.
Prove E_8 is positive-definite.

Definition 4.9

The E_8 sphere packing is the (periodic) sphere packing with separation $\sqrt{2}$, whose set of centres is Λ_8.

Lemma 4.10

$\text{Vol}(\Lambda_8) = \text{Covol}(\mathbb{R}^8/\Lambda_8) = 1$.

Proof for Lemma 4.10

TODO: In theory this should follow directly from $\det(\Lambda_8) = 1$, but Lean hates me and `EuclideanSpace` is being annoying.

Theorem 4.11

We have $\Delta_{\mathcal{P}(E_8)} = \frac{\pi^4}{384}$.

Proof for Theorem 4.11

By Theorem 3.5, we have $\Delta_{\mathcal{P}(E_8)} = |E_8/E_8| \cdot \frac{\text{Vol}\left(\mathcal{B}_8(\sqrt{2}/2)\right)}{\text{Covol}(E_8)} = \frac{\pi^4}{384}$, where the last equality follows from Lemma 4.10 and the formula for volume of a ball: $\text{Vol}(\mathcal{B}_d(R)) = R^d\pi^{d/2}/\Gamma\left(\frac{d}{2}+1\right)$.

5 Fourier Analysis

Recall the definition of a Fourier transform.
Definition 5.1

The Fourier transform of an L^1-function $f : \mathbb{R}^d \to \mathbb{C}$ is defined as

$$\mathcal{F}(f)(y) = \widehat{f}(y) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x,y \rangle} \, \mathrm{d}x, \quad y \in \mathbb{R}^d$$

where $\langle x, y \rangle = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x-y\|^2$ is the standard scalar product in $\mathbb{R}^d$.

The following computational result will be of use later on.
Lemma 5.2

$$\mathcal{F}(e^{\pi i \|x\|^2 z})(y) = z^{-4} \, e^{\pi i \|y\|^2 \, (\frac{-1}{z})}.$$

Proof for Lemma 5.2

Fill in proof.

Of great interest to us is a specific family of functions, known as Schwartz functions. The Fourier transform behaves particularly well when acting on Schwartz functions. We elaborate in the following subsections.
Definition 5.3

A C^∞ function $f : \mathbb{R}^d \to \mathbb{C}$ is called a Schwartz function if it decays to zero as $\|x\| \to \infty$ faster than any inverse power of $\|x\|$, and the same holds for all partial derivatives of f. That is, for all $k, n \in \mathbb{N}$, there exists a constant $C \in \mathbb{R}$ such that for all $x \in \mathbb{R}^d$, $\|x\|^k \cdot \left\| f^{(n)}(x) \right\| \leq C$, where $f^{(n)}$ denotes the n-th derivative of f together with the appropriate operator norm. The set of all Schwartz functions from $\mathbb{R}^d$ to $\mathbb{C}$ is called the Schwartz space. It is an $\mathbb{R}$-vector space.

Lemma 5.4

The Fourier transform is a continuous linear automorphism of the space of Schwartz functions.

Proof for Lemma 5.4

We do not elaborate here, as the result already exists in Mathlib. We do, however, mention that the Lean implementation defines a continuous linear equivalence on the Schwartz space using the Fourier transform. The proof that for any Schwartz function f, its Fourier transform and its image under this continuous linear equivalence are the same $\mathbb{R}^d \to \mathbb{R}$ function is stated in Mathlib solely for the purpose of `rw` and `simp`, and is proved simply by `rfl`.

Another reason we are interested in Schwartz functions is that they behave well under infinite sums. This will be useful when proving the Cohn-Elkies linear programming bound.

We begin by stating a general summability result over specific subsets of $\mathbb{R}^d$.

Lemma 5.5

Let $X \subset \mathbb{R}^d$ be a set of sphere-packing centers of separation 1 that is periodic with some lattice $\Lambda \subset \mathbb{R}^d$. Then there exists $k \in \mathbb{N}$ sufficiently large such that

$$\sum_{x \in X} \frac{1}{\|x\|^k} < \infty.$$

Proof for Lemma 5.5

First, note that it does not matter how we number the countably many elements of the discrete set X: if we prove absolute convergence for one numbering, we prove absolute convergence for any numbering. The idea is to bound the sequence of partial sums by considering the volumes of concentric d-spheres of the appropriate radii, or scaled versions of a zero-centered fundamental domain. /- source paragraph break -/ Finish.

The decaying property of Schwartz functions means that they can be compared with the absolutely convergent power series above.

Lemma 5.6

Let $f : \mathbb{R}^d \to \mathbb{C}$ be a Schwartz function and let $X \subset \mathbb{R}^d$ be periodic with respect to some lattice $\Lambda \subset \mathbb{R}^d$. Then

$$\sum_{x \in X} |f(x)| < \infty.$$

Proof for Lemma 5.6

Without loss of generality, assume that $0 \notin X$: if $0 \in X$, we can add the $f(0)$ term to the sum over nonzero elements of X. We know that for all $k \in \mathbb{N}$ there exists some constant C such that $|f(x)| \leq C \|x\|^{-k}$ for all $x \in \mathbb{R}^d$. Choosing k to be sufficiently large, we obtain

$$\sum_{x \in X} |f(x)| \leq \sum_{x \in X} \frac{C}{\|x\|^k} = C \sum_{x \in X} \frac{1}{\|x\|^k} < \infty.$$

We end with a crucial result on Schwartz functions, the statement of which only makes sense because of the above result.

Theorem 5.7

Let Λ be a lattice in $\mathbb{R}^d$, and let $f : \mathbb{R}^d \to \mathbb{R}$ be a Schwartz function. Then, for all $v \in \mathbb{R}^d$,

$$\sum_{\ell \in \Lambda} f(\ell + v) = \frac{1}{\text{Vol}(\mathbb{R}^d/\Lambda)} \sum_{m \in \Lambda^*} \widehat{f}(m) e^{-2\pi i \langle v, m \rangle}.$$

Proof for Theorem 5.7

One possible proof would be by induction on d. However, there are numerous nuances involved, particularly in manipulating nested infinite sums. Ideas would be appreciated.

While the Poisson summation formula over lattices can be stated in greater generality and probably should be formalised as such in Mathlib due to its many applications, we stick with Schwartz functions because that should be sufficient for our purposes.

Theorem 5.8

Assume $f : \mathbb{R} \to \mathbb{C}$ is smooth on $[0, \infty)$ and for all $k, n \in \mathbb{N}$ there exists $C \in \mathbb{R}$ such that

$$x^{\frac{k}{2}} \cdot |f^{(n)}(x)| \leq C.$$

Then, for all $d \in \mathbb{N}$, the function

$$f_d : \mathbb{R}^d \to \mathbb{C}, \quad f_d(x) := f(\|x\|^2)$$

is a Schwartz function.

6 Cohn-Elkies Bounds

In 2003 Cohn and Elkies developed linear programming bounds that apply directly to sphere packings; see Cohn and Elkies (2003). The goal of this section is to formalize the Cohn-Elkies linear programming bound.

The following theorem is the key result of Cohn and Elkies (2003). Note that the original theorem is stated for a class of functions more general then Schwartz functions.

Theorem 6.1

Let $X \subset \mathbb{R}^d$ be a discrete subset such that $\|x - y\| \geq 1$ for any distinct $x, y \in X$. Suppose that X is Λ-periodic with respect to some lattice $\Lambda \subset \mathbb{R}^d$. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a Schwartz function that is not identically zero and satisfies $f(x) \leq 0$ for $\|x\| \geq 1$ and $\widehat{f}(x) \geq 0$ for all $x \in \mathbb{R}^d$. Then the density of any Λ-periodic sphere packing is bounded above by

$$\frac{f(0)}{\widehat{f}(0)} \cdot \text{vol}(B_d(0, 1/2)).$$

Proof for Theorem 6.1

Here we reproduce the proof given in Cohn and Elkies (2003). The inequality

$$\sharp(X/\Lambda) \cdot f(0) \geq \sum_{x \in X} \sum_{y \in X/\Lambda} f(x - y) = \sum_{x \in X/\Lambda} \sum_{y \in X/\Lambda} \sum_{\ell \in \Lambda} f(x - y + l)$$

follows from the first Cohn-Elkies condition of the theorem and the assumption on the distances between points in X. /- -/ The equality

$$\sum_{x \in X/\Lambda} \sum_{y \in X/\Lambda} \sum_{\ell \in \Lambda} f(x-y+l) = \sum_{x \in X/\Lambda} \sum_{y \in X/\Lambda} \frac{1}{\text{vol}(\mathbb{R}^d/\Lambda)} \sum_{m \in \Lambda^*} \widehat{f}(m)\, e^{2\pi i m(x-y)}$$

follows from the Poisson summation formula. /- -/ The right-hand side of the above equation can be written as

$$\sum_{x \in X/\Lambda} \sum_{y \in X/\Lambda} \frac{1}{\text{vol}(\mathbb{R}^d/\Lambda)} \sum_{m \in \Lambda^*} \widehat{f}(m)\, e^{2\pi i m(x-y)} = \frac{1}{\text{vol}(\mathbb{R}^d/\Lambda)} \sum_{m \in \Lambda^*} \widehat{f}(m) \cdot \Big| \sum_{x \in X/\Lambda} e^{2\pi i m x} \Big|^2.$$

Note that $\left|\sum_{x\in X/\Lambda} e^{2\pi i m x}\right|^2 \geq 0$ for all $m \in \Lambda^*$. Moreover, the term corresponding to $m = 0$ satisfies $\left|\sum_{x\in X/\Lambda} e^{2\pi i 0 x}\right|^2 = \sharp(X/\Lambda)^2$.
/- -/ Now we use the second Cohn-Elkies condition and estimate

$$\frac{1}{\text{vol}(\mathbb{R}^d/\Lambda)} \sum_{m\in\Lambda^*} \widehat{f}(m) \cdot \Big|\sum_{x\in X/\Lambda} e^{2\pi i m(x-y)}\Big|^2 \geq \frac{\sharp(X/\Lambda)^2}{\text{vol}(\mathbb{R}^d/\Lambda)} \cdot \widehat{f}(0).$$

Comparing inequalities, we arrive at

$$\frac{\sharp(X/\Lambda)}{\text{vol}(\mathbb{R}^d/\Lambda)} \leq \frac{f(0)}{\widehat{f}(0)}.$$

Now we see that the density of the periodic packing $\mathcal{P}_X$ with balls of radius $1/2$ is bounded by

$$\Delta(\mathcal{P}_X) = \frac{\sharp(X/\Lambda)}{\text{vol}(\mathbb{R}^d/\Lambda)} \cdot \text{vol}(B_d(0,1/2)) \leq \frac{f(0)}{\widehat{f}(0)} \cdot \text{vol}(B_d(0,1/2)).$$

This finishes the proof of the theorem for periodic packings.

Theorem 6.2

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a Schwartz function that is not identically zero and satisfies Cohn-Elkies condition 1 and Cohn-Elkies condition 2 above. Then the density of any Λ-periodic sphere packing is bounded above by

$$\frac{f(0)}{\widehat{f}(0)} \cdot \text{vol}(B_d(0,1/2)).$$

Proof for Theorem 6.2

The result follows immediately from Theorem Theorem 2.13 and the Cohn-Elkies periodic theorem Theorem 6.1.

The main step in our proof of Corollary 2.16 is the explicit construction of an optimal function. It will be convenient for us to scale this function by $\sqrt{2}$.
Theorem 6.3

There exists a radial Schwartz function $g : \mathbb{R}^8 \to \mathbb{R}$ which satisfies:

$$g(x) \leq 0 \text{ for } \|x\| \geq \sqrt{2}$$

$$\widehat{g}(x) \geq 0 \text{ for all } x \in \mathbb{R}^8$$

$$g(0) = \widehat{g}(0) = 1.$$

7 Modular Forms

In this section, we recall and develop some theory of (quasi)modular forms.

Let $\mathbb{H}$ be the upper half-plane $\{z \in \mathbb{C} \mid \Im(z) > 0\}$.

Lemma 7.1

The modular group $\Gamma_1 := \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ by linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az+b}{cz+d}.$$

Proof for Lemma 7.1

Definition 7.2

The level N principal congruence subgroup of Γ_1 is

$$\Gamma(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1 \middle| \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \bmod N \right\}.$$

Definition 7.3

A subgroup $\Gamma \subset \Gamma_1$ is a congruence subgroup if $\Gamma(N) \subset \Gamma$ for some $N \in \mathbb{N}$.

Definition 7.4

Define the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \Gamma_1$$

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_1$$

$$\alpha = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \in \Gamma_2 \subset \Gamma_1$$

$$\beta = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in \Gamma_2 \subset \Gamma_1.$$

It is easily verifiable that $\alpha = T^2$ and $\beta = -S\alpha^{-1}S = -ST^{-2}S$.

Lemma 7.5

We have $\Gamma(1) = \langle S, T, -I \rangle$.

Lemma 7.6

We have $\Gamma(2) = \langle \alpha, \beta, -I \rangle$.

Let $z \in \mathbb{H}$, $k \in \mathbb{Z}$, and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. We omit many of the proofs below when they exist in Mathlib already.

Definition 7.7

The automorphy factor of weight k is defined as

$$j_k(z, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) := (cz + d)^{-k}.$$

Lemma 7.8

The automorphy factor satisfies the chain rule

$$j_k(z, \gamma_1\gamma_2) = j_k(z, \gamma_1)\, j_k(\gamma_2 z, \gamma_1).$$

Proof for Lemma 7.8

Definition 7.9

Let F be a function on $\mathbb{H}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Then the slash operator acts on F by

$$(F|_k\gamma)(z) := j_k(z, \gamma)\, F(\gamma z).$$

Lemma 7.10

The chain rule implies

$$F|_k\gamma_1\gamma_2 = (F|_k\gamma_1)|_k\gamma_2.$$

Proof for Lemma 7.10

Lemma 7.11

For even k, $F|_k(-I) = F$.

Proof for Lemma 7.11

Follows from the definition of the slash operator: $(F|_k(-I))(z) = (-1)^{-k}F((-I)z) = F(z)$.

Definition 7.12

Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A modular form of level Γ and weight $k \in \mathbb{Z}$ is a function $f : \mathbb{H} \to \mathbb{C}$ such that:

1. for all $\gamma \in \Gamma$, we have $f \mid_k \gamma = f$
2. f is holomorphic on $\mathbb{H}$
3. for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, there exist $A, B \in \mathbb{R}$ such that for all $z \in \mathbb{H}$ with $A \leq \mathrm{Im}(z)$, we have $|(f \mid_k \gamma)(z)| \leq B$. This defines a complex vector space denoted by $M_k(\Gamma)$.

Definition 7.13

For an even integer $k \geq 4$ define the weight-k Eisenstein series by

$$E_k(z) := \frac{1}{2} \sum_{(c,d)\in\mathbb{Z}^2,(c,d)=1} (cz+d)^{-k}.$$

Lemma 7.14

For all k, $E_k \in M_k(\Gamma_1)$. Especially, we have

$$E_k\left(-\frac{1}{z}\right) = z^k E_k(z).$$

Proof for Lemma 7.14

This follows from the fact that the sum converges absolutely. Applying the slash operator with $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ gives `eqn:Ek-trans-S`.

Lemma 7.15

Let Γ be a finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and let $f \in \mathcal{M}_k(\Gamma)$ be a modular form of weight k. Then the Fourier coefficients $a_n(f)$ have polynomial growth, i.e. $|a_n(f)| = O(n^k)$.

Proof for Lemma 7.15

Note that the assumption on polynomial growth holds when f is a holomorphic modular form, where the proof can be found in Serre (1973) for the case of level-one modular forms. This has been done in Lean 4 by David Loeffler.

Lemma 7.16

The Eisenstein series possesses the Fourier expansion

$$E_k(z) = 1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, e^{2\pi i z},$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. In particular,

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)\, e^{2\pi i n z}$$

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)\, e^{2\pi i n z}.$$

Proof for Lemma 7.16

Definition 7.17

Set

$$E_2(z) := 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)\, e^{2\pi i n z}.$$

Lemma 7.18

This function is not modular; however, it satisfies

$$z^{-2}\, E_2\left(-\frac{1}{z}\right) = E_2(z) - \frac{6i}{\pi}\,\frac{1}{z}.$$

Proof for Lemma 7.18

This is exercise 1.2.8 of Diamond and Shurman (2005).

Lemma 7.19

We have

$$(cz+d)^{-2} E_2\left(\frac{az+b}{cx+d}\right) = E_2(z) - \frac{6ic}{\pi(cz+d)}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Proof for Lemma 7.19

Use the fact that $\mathrm{SL}_2(\mathbb{Z})$ is generated by S and T. Then `eqn:E2-transform-general` follows from `eqn:E2-S-transform` together with $E_2|_T = E_2$.

Definition 7.20

The Dedekind eta function is defined as

$$\eta(z) = q^{1/24} \prod_{n \geq 1} (1 - q^n)$$

where $q = e^{2\pi i z}$.

Lemma 7.21

The Dedekind eta function transforms as

$$\eta\left(-\frac{1}{z}\right) = \sqrt{-iz}\eta(z).$$

Proof for Lemma 7.21

Consider the logarithmic derivative of η, which is equal to $\frac{\pi i}{12} E_2$. The result then follows from the transformation of E_2. See Diamond and Shurman (2005), Proposition `1.2.5`.

Definition 7.22

The discriminant form $\Delta(z)$ is given by

$$\Delta(z) = e^{2\pi i z} \prod_{n \geq 1} (1 - e^{2\pi i n z})^{24}.$$

Lemma 7.23

$\Delta(z) \in M_{12}(\Gamma_1)$. Especially,

$$\Delta\left(-\frac{1}{z}\right) = z^{12}\Delta(z).$$

Also, it vanishes at the unique cusp, i.e. it is a cusp form of level Γ_1 and weight 12.

Proof for Lemma 7.23

The fact that it is invariant under translation is clear from the definition, so we only need to check transformation under S. Now, note that $\eta^{24} = \Delta$, and from Lemma 7.21 we have $\eta(-1/z) = \sqrt{-iz}\eta(z)$, so $\Delta(-1/z) = z^{12}\Delta(z)$ as required.

Lemma 7.24

We have

$$\Delta(z) = (E_4^3 - E_6^2)/1728.$$

Proof for Lemma 7.24

We only need to show its a cuspform, since once we have this, dividing the rhs by Δ would give a modular form of weight 0 which is a constant, and so we can determine the constant easily. /- source paragraph break -/ To check its a cuspform, we just look at the q-expansions of E_4 and E_6 and prove directly that the first term vanishes.

Corollary 7.25

$\Delta(it) > 0$ for all $t > 0$.

Proof for Corollary 7.25

By Definition 7.22,

$$\Delta(it) = e^{-2\pi t} \prod_{n \geq 1} (1 - e^{-2\pi n t})^{24} > 0.$$

Corollary 7.26

$\Delta(z) \neq 0$ for all $z \in \mathfrak{H}$.

Proof for Corollary 7.26

This follows from the product formula.

Theorem 7.27

Let $k \in \mathbb{Z}$ with $k < 0$. Then $M_k(\Gamma_1) = \{0\}$ and moreover $\dim M_0(\Gamma(1)) = 1$.

Proof for Theorem 7.27

The proof makes use of the maximum modulus principle. Since this is already formalized, we skip the details here, but see the Lean proof for them.

Theorem 7.28

Let $k \in \mathbb{Z}$ with $k \geq 0$ and even. Then $\dim M_k(\Gamma_1) = \lfloor k/12 \rfloor$ if $k \equiv 2 \mod 12$, and $\dim M_k(\Gamma_1) = \lfloor k/12 \rfloor + 1$ if $k \not\equiv 2 \mod 12$.

Proof for Theorem 7.28

First we note that for $2 < k$ we have $\dim(M_k(\Gamma_1)) = 1 + \dim S_k(\Gamma_1)$. This follows since we know the E_k are in M_k so by scaling appropriately, any non-cuspform $f \in M_k$ we would have $f - aE_k \in S_k$ for some a. /- source paragraph break -/ Next, note that $S_k(\Gamma_1)$ is isomorphic to $M_{k-12}(\Gamma_1)$, since if $f \in S_k$ then f/Δ is now a modular form (using the product expansion of Δ and its non-vanishing on $\mathfrak{H}$) of weight $k - 12$. Note its important that f is a cuspform so that the quotient by Δ is a modular form. /- source paragraph break -/ So we only need to know the dimensions of $M_k(\Gamma_1)$ for $0 \leq k \leq 12$. For $k = 0$ we have $\dim M_0(\Gamma_1) = 1$ by Theorem 7.27. For $k = 4$ we have $\dim M_4(\Gamma_1) = 1$ since if there was a cuspform f of weight 4 then f/Δ would be a modular form of negative weight, i.e. zero, so $f = 0$. Similarly for $k = 6, 8, 10$. For $k = 12$ we have $\dim S_{12}(\Gamma_1) = 1$ since the discriminant form is a cusp form of weight 12 and any other cusp form of weight 12 would be a scalar multiple of Δ (since their ratio would be a modular form of weight 0). So we have $\dim M_{12}(\Gamma_1) = 2$. /- source paragraph break -/ Finally we need to check that $\dim M_2(\Gamma_1) = 0$. Firstly, there can't be any cuspforms here by the same argument as above. So we need to check that there are no modular forms of weight 2. If we did have one, call it f then f^2 would be a non-cuspform of weight 4 and so $f^2 = aE_4$, where in fact $a = a_0(f)^2$ (since $(f^2 - a_0(f)E_4)$ is now a cuspform of weight 4 which means its zero). Similarly, $f^3 = a_0(f)^3 E_6$. But now taking powers to make them weight 12 forms we see that $a_0(f)^6(E_4^3 - E_6^2) = 0 = 1728 a_0(f)^6 \Delta$, but $a_0(f) \neq 0$ (since its assumed to not be a cuspform), this would mean $\Delta = 0$ which we know can't happen.

Theorem 7.29

Let Γ be a congruence subgroup. Then $M_k(\Gamma)$ is finite-dimensional.

Proof for Theorem 7.29

We know that $\dim(M_k(\Gamma_1))$ is finite-dimensional, hence there is some r_k such that any element of $M_k(\Gamma_1)$ vanishing at infinity to degree greater than r_k must be zero. Now take $f \in M_k(\Gamma)$ vanishing to degree n at infinity, and set $F = \prod_\gamma f \mid_k \gamma$, where the product is over a set of representatives of $\Gamma_1 \backslash \Gamma$. Then F is a modular form of weight kd where $d = [\Gamma_1 : \Gamma]$, and it vanishes at infinity to degree at least n. So if $n > r_{kd}$, then $F = 0$, and hence $f = 0$.

Corollary 7.30

We have

$$\dim M_2(\mathrm{SL}_2(\mathbb{Z})) = 0$$

$$\dim M_4(\mathrm{SL}_2(\mathbb{Z})) = \dim M_6(\mathrm{SL}_2(\mathbb{Z})) = \dim M_8(\mathrm{SL}_2(\mathbb{Z})) = 1$$
$$\dim S_4(\mathrm{SL}_2(\mathbb{Z})) = \dim S_6(\mathrm{SL}_2(\mathbb{Z})) = \dim S_8(\mathrm{SL}_2(\mathbb{Z})) = 0.$$

Proof for Corollary 7.30

Definition 7.31

We define three different theta functions, the "Thetanullwerte", $\Theta_2, \Theta_3, \Theta_4$, by

$$\Theta_2(z) = \theta_{10}(z) = \sum_{n\in\mathbb{Z}} e^{\pi i(n+\frac{1}{2})^2 z}$$

$$\Theta_3(z) = \theta_{00}(z) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}$$

$$\Theta_4(z) = \theta_{01}(z) = \sum_{n\in\mathbb{Z}} (-1)^n\, e^{\pi i n^2 z}.$$

Definition 7.32

Define $H_2 = \Theta_2^4$, $H_3 = \Theta_3^4$, $H_4 = \Theta_4^4$.

Lemma 7.33

These elements act on the theta functions in the following way.

$$H_2|S = -H_4, \quad H_3|S = -H_3, \quad H_4|S = -H_2$$

and

$$H_2|T = -H_2, \quad H_3|T = H_4, \quad H_4|T = H_3.$$

Proof for Lemma 7.33

The last three identities easily follow from the definition. For example, `eqn:H2-transform-T` follows from

$$\Theta_2(z+1) = \sum_{n\in\mathbb{Z}} e^{\pi i(n+\frac{1}{2})^2(z+1)} = \sum_{n\in\mathbb{Z}} e^{\pi i(n+\frac{1}{2})^2} e^{\pi i(n+\frac{1}{2})^2 z}$$

$$= \sum_{n\in\mathbb{Z}} e^{\pi i(n^2+n+\frac{1}{4})} e^{\pi i(n+\frac{1}{2})^2 z} = \sum_{n\in\mathbb{Z}} (-1)^{n^2+n} e^{\pi i/4} e^{\pi i(n+\frac{1}{2})^2 z}$$

$$= e^{\pi i/4}\Theta_2(z),$$

and taking fourth powers. The identities `eqn:H2-transform-S` and `eqn:H4-transform-S` are equivalent under $z \leftrightarrow -1/z$, so it is enough to show `eqn:H2-transform-S` and `eqn:H3-transform-S`. These

identities follow from the identities of the two-variable Jacobi theta function, which is defined as

$$\theta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{2\pi i n z + \pi i n^2 \tau}.$$

This function specializes to the theta functions as

$$\Theta_2(\tau) = e^{\pi i \tau/4}\theta(-\tau/2,\tau),$$

$$\Theta_3(\tau) = \theta(0,\tau),$$

$$\Theta_4(\tau) = \theta(1/2,\tau),$$

and Poisson summation gives

$$\theta(z,\tau) = \frac{1}{\sqrt{-i\tau}} e^{-\frac{\pi i z^2}{\tau}} \theta\left(\frac{z}{\tau}, -\frac{1}{\tau}\right).$$

Applying these specializations yields the identities. For example, `eqn:H4-transform-S` follows from

$$\Theta_4(\tau) = \theta\left(\frac{1}{2},\tau\right) = \frac{1}{\sqrt{-i\tau}} e^{-\frac{\pi i}{4\tau}} \theta\left(\frac{1}{2\tau}, -\frac{1}{\tau}\right) = \frac{1}{\sqrt{-i\tau}} \Theta_2\left(-\frac{1}{\tau}\right),$$

and taking fourth powers.

Lemma 7.34

H_2, H_3, and H_4 are slash invariant under $\Gamma(2)$, that is, for all $\gamma \in \Gamma(2)$ and $i \in \{2,3,4\}$, we have $H_i|\gamma = H_i|\gamma^{-1} = H_i$.

Proof for Lemma 7.34

By Lemma 7.6 and Lemma 7.10, it suffices to show that the H_i are invariant under slash actions with respect to α, β, and $-I$. Invariance under $-I$ follows from Lemma 7.11. The rest follows from Lemma 7.10, Lemma 7.33, and the matrix identities $\alpha = T^2$ and $\beta = -S\alpha^{-1}S = -ST^{-2}S$. For example, invariance for H_2 follows from $H_2|\alpha = H_2|T^2 = -H_2|T = H_2$ and $H_2|\beta = H_2|(-S\alpha^{-1}S) = H_2|(S\alpha^{-1}S) = -H_4|(\alpha^{-1}S) = -H_4|S = H_2$.

Lemma 7.35

For all $\gamma \in \Gamma_1$, the slash-translates $H_2|_2\gamma$, $H_3|_2\gamma$, and $H_4|_2\gamma$ are holomorphic at $i\infty$.

Proof for Lemma 7.35

We want to show that for $\gamma \in \Gamma_1$, $\|H_2|_2\gamma(z)\|$ is bounded as $z \in \mathbb{H} \to i\infty$. By Lemma 7.33, Lemma 7.6, and induction on group elements, the set $\{\pm H_2, \pm H_3, \pm H_4\}$ is closed under the action of Γ_1. Hence it suffices to prove that H_2, H_3, H_4 are bounded at $i\infty$. For $z \in \mathbb{H}$ with $\Im(z) \geq A$,

$$\|H_2(z)\| = \left\| \sum_{n\in\mathbb{Z}} \exp\left(\pi i \left(n+\frac{1}{2}\right)^2 z\right) \right\|^4$$

$$\leq \left(\sum_{n\in\mathbb{Z}} \left\| \exp\left(\pi i \left(n+\frac{1}{2}\right)^2 z\right) \right\| \right)^4 = \left(\sum_{n\in\mathbb{Z}} \left\| \exp\left(-\pi \left(n+\frac{1}{2}\right)^2 \Im(z)\right) \right\| \right)^4$$

$$\leq \left(\sum_{n\in\mathbb{Z}} \left\| \exp\left(-\pi \left(n+\frac{1}{2}\right)^2 A\right) \right\| \right)^4.$$

The final term is convergent because it equals $\exp(-\pi A/4)\theta(iA/2, iA)$. The proofs for H_3 and H_4 are similar.

Lemma 7.36

H_2, H_3, and H_4 belong to $M_2(\Gamma(2))$.

Proof for Lemma 7.36

From Lemma 7.34 and Lemma 7.35, it remains ot prove that H_2, H_3 and H_4 are holomorphic on $\mathbb{H}$. fill in proof.

Lemma 7.37

H_2 admits a Fourier series of the form

$$H_2(z) = \sum_{n\geq 1} c_{H_2}(n) e^{\pi i n z}$$

for some $c_{H_2}(n) \in \mathbb{R}_{\geq 0}$, with $c_{H_2}(1) = 16$ and $c_{H_2}(n) = O(n^k)$ for some $k \in \mathbb{N}$.

Proof for Lemma 7.37

We have $H_2(z) = \Theta_2(z)^4 = \left(\sum_{n\in\mathbb{Z}} e^{\pi i (n+\frac{1}{2})^2 z}\right)^4 = \left(2\sum_{n\geq 0} e^{\pi i (n+\frac{1}{2})^2 z}\right)^4$, and then $\left(2e^{\pi i z/4} + 2\sum_{n\geq 1} e^{\pi i (n^2+n+\frac{1}{4})z}\right)^4 = 16e^{\pi i z}\left(1 + \sum_{n\geq 1} e^{\pi i (n^2+n) z}\right)^4$, so $H_2(z) = 16e^{\pi i z} + \sum_{n\geq 2} c_{H_2}(n) e^{\pi i n z} = \sum_{n\geq 1} c_{H_2}(n) e^{\pi i n z}$.

Lemma 7.38

H_3 admits a Fourier series of the form

$$H_3(z) = \sum_{n \geq 0} c_{H_3}(n) e^{\pi i n z}$$

for some $c_{H_3}(n) \in \mathbb{R}_{\geq 0}$ with $c_{H_3}(0) = 1$ and $c_{H_3}(n) = O(n^k)$ for some $k \in \mathbb{N}$. Especially, H_3 is not cuspidal.

Proof for Lemma 7.38

We have $H_3(z) = \Theta_3(z)^4 = \left(\sum_{n \in \mathbb{Z}} e^{\pi i n^2 z}\right)^4 = \left(1 + 2\sum_{n \geq 1} e^{\pi i n^2 z}\right)^4 = 1 + O(e^{\pi i z})$.

Lemma 7.39

H_4 admits a Fourier series of the form

$$H_4(z) = \sum_{n \geq 0} c_{H_4}(n) e^{\pi i n z}$$

for some $c_{H_4}(n) \in \mathbb{R}$ with $c_{H_4}(0) = 1$ and $c_{H_4}(n) = O(n^k)$ for some $k \in \mathbb{N}$. Especially, H_4 is not cuspidal.

Lemma 7.40

These three theta functions satisfy the Jacobi identity

$$H_2 + H_4 = H_3 \Leftrightarrow \Theta_2^4 + \Theta_4^4 = \Theta_3^4.$$

Proof for Lemma 7.40

Let $f = (H_2 + H_4 - H_3)^2$. Obviously, f is a modular form of weight 4 and level $\Gamma(2)$. Using the transformation rules of H_2, H_3, H_4, we have $f|_S = (-H_4 - H_2 + H_3)^2 = f$ and $f|_T = (-H_2 + H_3 - H_4)^2 = f$, so f is actually a modular form of level 1. By considering the limit as $z \to i\infty$, f is a cusp form, and hence $f = 0$ by `eqn:dimS4`.

Lemma 7.41

We have

$$E_4 = \frac{1}{2}(H_2^2 + H_3^2 + H_4^2) = H_2^2 + H_2 H_4 + H_4^2$$

$$E_6 = \frac{1}{2}(H_2 + H_3)(H_3 + H_4)(H_4 - H_2) = \frac{1}{2}(H_2 + 2H_4)(2H_2 + H_4)(H_4 - H_2)$$

$$\Delta = \frac{1}{256}(H_2 H_3 H_4)^2.$$

Proof for Lemma 7.41

We can prove these similarly as Lemma Lemma 7.40. The right-hand sides of `eqn:e4theta`, `eqn:e6theta`, and `eqn:disctheta` are all modular forms of level Γ_1 and of the desired weights, where `eqn:disctheta` is a cusp form since H_2 is. Now the identities follow from the dimension calculations $\dim M_4(\Gamma_1) = \dim M_6(\Gamma_1) = \dim S_{12}(\Gamma_1) = 1$ and comparing the first nonzero q-coefficients.

Corollary 7.42

$H_2(it)$ and $H_4(it)$ are positive for $t > 0$.

Proof for Corollary 7.42

By the transformation law `eqn:H2-transform-S`, it is enough to prove the positivity for $\Theta_2(it)$, which is clear from its definition:

$$\Theta_2(it) = \sum_{n\in\mathbb{Z}} e^{-\pi(n+\frac{1}{2})^2 t} > 0.$$

Definition 7.43

Let F be a quasimodular form. We define the (normalized) derivative of F as

$$F' = DF := \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}z}F.$$

Lemma 7.44

We have an equality of operators $D = q\frac{\mathrm{d}}{\mathrm{d}q}$. In particular, if $F(z) = \sum_{n\geq n_0} a_n q^n$, then $F'(z) = \sum_{n\geq n_0} n a_n q^n$.

Proof for Lemma 7.44

This follows directly from the definition `def:derivative`, since $\frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}z}e^{2\pi i n z} = n e^{2\pi i n z}$.

Definition 7.45

For $k \in \mathbb{R}$, define the weight-k Serre derivative ∂_k of a modular form F by

$$\partial_k F := F' - \frac{k}{12}E_2 F.$$

Theorem 7.46

Serre derivative ∂_k is equivariant with the slash action of $\mathrm{SL}_2(\mathbb{Z})$ in the sense that

$$\partial_k(F|_k\gamma) = (\partial_k F)|_{k+2}\gamma, \quad \forall \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

Proof for Theorem 7.46

Let $G = \partial_k F = F' - \frac{k}{12}E_2 F$. From $F \in M_k(\Gamma)$, we have

$$(F|_k\gamma)(z) := (cz+d)^{-k} F\left(\frac{az+b}{cz+d}\right), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

By taking the derivative of the above equation, we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}(F|_k\gamma)(z) &= -kc(cz+d)^{-k-1} F\left(\frac{az+b}{cz+d}\right) + (cz+d)^{-k}(cz+d)^{-2}\frac{\mathrm{d}F}{\mathrm{d}z}\left(\frac{az+b}{cz+d}\right) \\
&= -kc(cz+d)^{-k-1} F\left(\frac{az+b}{cz+d}\right) + (cz+d)^{-k-2}\frac{\mathrm{d}F}{\mathrm{d}z}\left(\frac{az+b}{cz+d}\right) \\
&= -kc(cz+d)^{-k-1} F\left(\frac{az+b}{cz+d}\right) + 2\pi i(cz+d)^{-k-2} F'\left(\frac{az+b}{cz+d}\right) \\
\Leftrightarrow (F|_k\gamma)'(z) &= -\frac{kc}{2\pi i}(cz+d)^{-k-1} F\left(\frac{az+b}{cz+d}\right) + (cz+d)^{-k-2} F'\left(\frac{az+b}{cz+d}\right).
\end{aligned}$$

Combined with `eqn:E2-transform-general`, we get

$$\begin{aligned}
((\partial_k F)|_{k+2}\gamma)(z) &= (cz+d)^{-k-2}\left(F'\left(\frac{az+b}{cz+d}\right) - \frac{k}{12}E_2\left(\frac{az+b}{cz+d}\right) F\left(\frac{az+b}{cz+d}\right)\right) \\
&= (cz+d)^{-k-2} F'\left(\frac{az+b}{cz+d}\right) - \frac{k}{12}\left(E_2(z) - \frac{6ic}{\pi(cz+d)}\right)\cdot(cz+d)^{-k} F\left(\frac{az+b}{cz+d}\right) \\
&= (F|_k\gamma)'(z) - \frac{k}{12}E_2(z)(F|_k\gamma)(z) \\
&= \partial_k(F|_k\gamma)(z).
\end{aligned}$$

Theorem 7.47

Let F be a modular form of weight k and level Γ. Then $\partial_k F$ is a modular form of weight $k+2$ of the same level.

Proof for Theorem 7.47

Immediate from Theorem Theorem 7.46 since $F|_k\gamma = F$ for all $\gamma \in \Gamma$.

Theorem 7.48

We have

$$E_2' = \frac{E_2^2 - E_4}{12}$$
$$E_4' = \frac{E_2 E_4 - E_6}{3}$$
$$E_6' = \frac{E_2 E_6 - E_4^2}{2}.$$

Proof for Theorem 7.48

In terms of Serre derivatives, these are equivalent to

$$\partial_1 E_2 = -\frac{1}{12} E_4$$
$$\partial_4 E_4 = -\frac{1}{3} E_6$$
$$\partial_6 E_6 = -\frac{1}{2} E_4^2.$$

By Theorem Theorem 7.47, all the Serre derivatives are, in fact, modular. To be precise, the modularity of $\partial_4 E_4$ and $\partial_6 E_6$ directly follows from Theorem Theorem 7.47, and that of $\partial_1 E_2$ follows from `eqn:E2-transform-general`. Differentiating and squaring then gives us the following:

$$E_2'|_4\gamma = E_2' - \frac{ic}{\pi(cz+d)} E_2 - \frac{3c^2}{\pi^2(cz+d)^2}$$
$$E_2^2|_4\gamma = E_2^2 - \frac{12ic}{\pi(cz+d)} E_2 - \frac{36c^2}{\pi^2(cz+d)^2}.$$

Hence, `eqn:DE2` $-\frac{1}{12}$ `eqn:E2sq-transform` is a modular form of weight 4. By Corollary 7.30, they should be multiples of E_4, E_6, E_4^2, and the proportionality constants can be determined by observing the constant terms of q-expansions.

Corollary 7.49

We have

$$\Delta' = E_2 \Delta.$$

Proof for Corollary 7.49

By Ramanujan's formula `eqn:DE4` and `eqn:DE6`,

$$\Delta' = \frac{3E_4^2 E_4' - 2E_6 E_6'}{1728} = \frac{1}{1728}\left(3E_4^2 \cdot \frac{E_2 E_4 - E_6}{3} - 2E_6 \cdot \frac{E_2 E_6 - E_4^2}{2}\right) = \frac{E_2(E_4^3 - E_6^2)}{1728} = E_2 \Delta.$$

Lemma 7.50

We have

$$H_2' = \frac{1}{6}(H_2^2 + 2H_2H_4 + E_2H_2)$$

$$H_3' = \frac{1}{6}(H_2^2 - H_4^2 + E_2H_3)$$

$$H_4' = -\frac{1}{6}(2H_2H_4 + H_4^2 - E_2H_4)$$

or equivalently,

$$\partial_2 H_2 = \frac{1}{6}(H_2^2 + 2H_2H_4)$$

$$\partial_2 H_3 = \frac{1}{6}(H_2^2 - H_4^2)$$

$$\partial_2 H_4 = -\frac{1}{6}(2H_2H_4 + H_4^2).$$

Proof for Lemma 7.50

Equivalences are obvious from the definition of the Serre derivative. Define f_2, f_3, f_4 be the differences of the left and right hand sides of `eqn:H2-serre-der`, `eqn:H3-serre-der`, `eqn:H4-serre-der`.

$$f_2 := \partial_2 H_2 - \frac{1}{6}H_2(H_2 + 2H_4)$$

$$f_3 := \partial_2 H_3 - \frac{1}{6}(H_2^2 - H_4^2)$$

$$f_4 := \partial_2 H_4 + \frac{1}{6}H_4(2H_2 + H_4).$$

Then these are a priori modular forms of weight 4 and level $\Gamma(2)$, and our goal is to prove that they are actually zeros. By Jacobi's identity `eqn:jacobi-identity`, we have $f_2 + f_4 = f_3$. Also, the transformation rules of H_2, H_3, H_4 give

$$f_2|_S = -f_4$$

$$f_2|_T = -f_2$$

$$f_4|_S = -f_2$$

$$f_4|_T = f_3 = f_2 + f_4.$$

Now define

$$g := (2H_2 + H_4)f_2 + (H_2 + 2H_4)f_4$$

$$h := f_2^2 + f_2f_4 + f_4^2.$$

Then one can check that both g and h are invariant under the actions of S and T, hence they are modular forms of level 1. Also, by analyzing the limit of g and h as $z \to i\infty$, one can see that g and h are cusp forms, hence $g = h = 0$ by `eqn:dimS6` and `eqn:dimS8`. This implies

$$3E_4 f_2^2 = 3(H_2^2+H_2H_4+H_4^2)f_2^2 = ((2H_2+H_4)^2-(2H_2+H_4)(H_2+2H_4)+(H_2+2H_4)^2)f_2^2$$
$$= (2H_2 + H_4)^2(f_2^2 + f_2f_4 + f_4^2) = 0$$

and by considering q-series (E_4 has an invertible q-series), we get $f_2 = 0$.

Theorem 7.51

The Serre derivative satisfies the following product rule: for any quasimodular forms F and G,

$$\partial_{w_1+w_2}(FG) = (\partial_{w_1}F)G + F(\partial_{w_2}G).$$

Proof for Theorem 7.51

It follows from the definition:

$$\partial_{w_1+w_2}(FG) = (FG)' - \frac{w_1+w_2}{12}E_2(FG)$$
$$= F'G + FG' - \frac{w_1+w_2}{12}E_2(FG)$$
$$= \left(F' - \frac{w_1}{12}E_2F\right)G + F\left(G' - \frac{w_2}{12}E_2G\right)$$
$$= (\partial_{w_1}F)G + F(\partial_{w_2}G).$$

We also have the following useful theorem for proving positivity of quasimodular forms on the imaginary axis, which is Proposition 3.5 and Corollary 3.6 of Lee (2024).

Theorem 7.52

Let F be a holomorphic quasimodular cusp form with real Fourier coefficients. Assume that there exists k such that $(\partial_k F)(it) > 0$ for all $t > 0$. If the first Fourier coefficient of F is positive, then $F(it) > 0$ for all $t > 0$.

Proof for Theorem 7.52

By `eqn:logder-disc-E2` we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{F(it)}{\Delta(it)^{\frac{k}{12}}}\right) = (-2\pi)\frac{F'(it)\Delta(it)^{\frac{k}{12}} - F(it)\frac{k}{12}E_2(it)\Delta(it)^{\frac{k}{12}}}{\Delta(it)^{\frac{k}{6}}} = (-2\pi)\frac{(\partial_k F)(it)}{\Delta(it)^{\frac{k}{12}}} < 0.$$

Hence the function $t \mapsto \frac{F(it)}{\Delta(it)^{\frac{k}{12}}}$ is monotone decreasing. Because the first nonzero Fourier coefficient of F is positive, $F(it) > 0$ for sufficiently large t:

$$F = \sum_{n \geq n_0} a_n q^n \Rightarrow e^{2\pi n_0 t}F(it) = a_{n_0} + e^{-2\pi t}\sum_{n \geq n_0+1} a_n e^{-2\pi(n-n_0-1)t},$$

and $\lim_{t\to\infty} e^{2\pi n_0 t}F(it) = a_{n_0} > 0$. The result follows.

8 Fourier Eigenfunctions with Double Zeroes

First we define the function a. To this end, we consider the following functions.

Definition 8.1

Define

$$\phi_{-4} := \frac{E_4^2}{\Delta}$$
$$\phi_{-2} := \frac{E_4(E_2E_4 - E_6)}{\Delta}$$
$$\phi_0 := \frac{(E_2E_4 - E_6)^2}{\Delta}.$$

The function $\phi_0(z)$ is not modular; however, it satisfies the following transformation rules.

Lemma 8.2

We have

$$\phi_0(z+1) = \phi_0(z)$$
$$\phi_0\left(-\frac{1}{z}\right) = \phi_0(z) - \frac{12i}{\pi}\frac{1}{z}\phi_{-2}(z) - \frac{36}{\pi^2}\frac{1}{z^2}\phi_{-4}(z).$$

Proof for Lemma 8.2

Equation $\phi_0(z+1) = \phi_0(z)$ follows easily from periodicity of the Eisenstein series and of $\Delta(z)$. For the second transformation law,

$$\phi_0\left(-\frac{1}{z}\right) = \frac{(E_2(-1/z)E_4(-1/z) - E_6(-1/z))^2}{\Delta(-1/z)}$$
$$= \frac{((z^2E_2(z) - 6iz/\pi)\cdot z^4E_4(z) - z^6E_6(z))^2}{z^{12}\Delta(z)}$$

$$= \frac{\left(E_2(z)E_4(z) - E_6(z) - \frac{6i}{\pi z}E_4(z)\right)^2}{\Delta(z)}$$

$$= \frac{(E_2(z)E_4(z) - E_6(z))^2 - \frac{12i}{\pi z}(E_2(z)E_4(z) - E_6(z))E_4(z) - \frac{36}{\pi^2 z^2}E_4(z)^2}{\Delta(z)}$$

$$= \phi_0(z) - \frac{12i}{\pi z}\phi_{-2}(z) - \frac{36}{\pi^2 z^2}\phi_{-4}(z).$$

For our formalization, we choose rectangular contours for the first and second integrals, and write the definition as follows.

Definition 8.3

Define $a_{\text{rad}} : \mathbb{R} \to \mathbb{C}$ by

$$a_{\text{rad}}(r) := I_1(r) + I_2(r) + I_3(r) + I_4(r) + I_5(r) + I_6(r)$$

where

$$I_1(r) := \int_{-1}^{-1+i} \phi_0\left(\frac{-1}{z+1}\right)(z+1)^2 e^{\pi i r z}\mathrm{d}z$$

$$I_2(r) := \int_{-1+i}^{i} \phi_0\left(\frac{-1}{z+1}\right)(z+1)^2 e^{\pi i r z}\mathrm{d}z$$

$$I_3(r) := \int_{1}^{1+i} \phi_0\left(\frac{-1}{z-1}\right)(z-1)^2 e^{\pi i r z}\mathrm{d}z$$

$$I_4(r) := \int_{1+i}^{i} \phi_0\left(\frac{-1}{z-1}\right)(z-1)^2 e^{\pi i r z}\mathrm{d}z$$

$$I_5(r) := -2\int_{0}^{i} \phi_0\left(\frac{-1}{z}\right) z^2 e^{\pi i r z}\mathrm{d}z$$

$$I_6(r) := 2\int_{i}^{i\infty} \phi_0(z) e^{\pi i r z}\mathrm{d}z.$$

Here all contours are chosen to be straight line segments. Define $a(x) := a_{\text{rad}}(\|x\|^2)$ for $x \in \mathbb{R}^8$.

In the original paper, the integrals I_1 and I_2 are combined, as are I_3 and I_4. We write them separately to simplify the formalization.

Now we prove that a satisfies condition `eqn:a-fourier`. The following lemma will be used to prove Schwartzness of both a and b.

Lemma 8.4

Let $f(z)$ be a holomorphic function with Fourier expansion

$$f(z) = \sum_{n \geq n_0} c_f(n) e^{\pi i n z}$$

with $c_f(n_0) \neq 0$. Assume that $c_f(n)$ has polynomial growth, that is, $|c_f(n)| = O(n^k)$ for some $k \in \mathbb{N}$. Then there exists a constant $C_f > 0$ such that

$$\left|\frac{f(z)}{\Delta(z)}\right| \leq C_f e^{-\pi(n_0-2)\Im z}$$

for all z with $\Im z > 1/2$.

When $f(z)$ is a quasi-modular form, we can take k to be the weight of f.

Proof for Lemma 8.4

By the product formula for Δ,

$$\left|\frac{f(z)}{\Delta(z)}\right| = \left|\frac{\sum_{n\geq n_0} c_f(n)e^{\pi inz}}{e^{2\pi iz}\prod_{n\geq 1}(1-e^{2\pi inz})^{24}}\right|$$

$$= |e^{\pi i(n_0-2)z}| \cdot \frac{|\sum_{n\geq n_0} c_f(n)e^{\pi i(n-n_0)z}|}{\prod_{n\geq 1}|1-e^{2\pi inz}|^{24}}$$

$$\leq e^{-\pi(n_0-2)\Im z} \cdot \frac{\sum_{n\geq n_0}|c_f(n)|e^{-\pi(n-n_0)\Im z}}{\prod_{n\geq 1}(1-e^{-2\pi n\Im z})^{24}}$$

$$\leq e^{-\pi(n_0-2)\Im z} \cdot \frac{\sum_{n\geq n_0}|c_f(n)|e^{-\pi(n-n_0)/2}}{\prod_{n\geq 1}(1-e^{-\pi n})^{24}} = C_f \cdot e^{-\pi(n_0-2)\Im z}.$$

Here

$$C_f = \frac{\sum_{n\geq n_0}|c_f(n)|e^{-\pi(n-n_0)/2}}{\prod_{n\geq 1}(1-e^{-\pi n})^{24}}.$$

The numerator converges absolutely because of polynomial growth, and the denominator also converges; indeed it is simply $e^{\pi}\cdot\Delta(i/2)$.

As corollaries, we obtain the following bounds for ϕ_0, ϕ_{-2}, and ϕ_{-4}.

Corollary 8.5

There exists a constant $C_0 > 0$ such that

$$|\phi_0(z)| \leq C_0 e^{-2\pi\Im z}$$

for all z with $\Im z > 1/2$.

Proof for Corollary 8.5

By Ramanujan's formula, $E_2E_4 - E_6 = 3E_4' = 720\sum_{n\geq 1} n\sigma_3(n)e^{2\pi inz}$, and

$$(E_2(z)E_4(z) - E_6(z))^2 = 720^2 e^{4\pi iz} + O(e^{5\pi iz}).$$

The result then follows from Lemma Lemma 8.4 with $f(z) = (E_2E_4 - E_6)^2$ and $n_0 = 4$.

Corollary 8.6

There exists a constant $C_{-2} > 0$ such that

$$|\phi_{-2}(z)| \leq C_{-2}$$

for all z with $\Im z > 1/2$.

Corollary 8.7

There exists a constant $C_{-4} > 0$ such that

$$|\phi_{-4}(z)| \leq C_{-4} e^{2\pi \Im z}$$

for all z with $\Im z > 1/2$.

Note that we can take the constants C_0, C_{-2}, and C_{-4} as

$$C_0 = 9 \cdot 240^2 \cdot e^{\pi} \cdot \frac{E_4'(i/2)^2}{\Delta(i/2)}$$

$$C_{-2} = 3 \cdot \frac{E_4(i/2) E_4'(i/2)}{\Delta(i/2)}$$

$$C_{-4} = e^{-\pi} \cdot \frac{E_4(i/2)^2}{\Delta(i/2)}.$$

Lemma 8.8

For all $n \in \mathbb{N}$, there exists a constant C' such that for all $r \geq 0$,

$$r^n \cdot \int_1^{\infty} e^{-2\pi s}\, e^{-\pi r/s}\, \mathrm{d}s \leq C'.$$

Proof for Lemma 8.8

Fix $n \in \mathbb{N}$. There exists a constant C such that for all $x \geq 0$, $|x|^n \cdot |e^{-\pi x}| \leq C$. In particular, for all $r \geq 0$ and $s \geq 1$, $r^n \cdot e^{-\pi r/s} \leq C s^n$. Hence

$$r^n \cdot \int_1^{\infty} e^{-2\pi s}\, e^{-\pi r/s}\, \mathrm{d}s = \int_1^{\infty} e^{-2\pi s} \left(|r|^n \cdot e^{-\pi r/s} \right) \mathrm{d}s \leq C \int_1^{\infty} e^{-2\pi s}\, s^n\, \mathrm{d}s.$$

The Gamma function is given by

$$\Gamma(x) = \int_0^{\infty} e^{-u}\, u^{x-1}\, \mathrm{d}u$$

for all $x > 0$. Writing $u = 2\pi s$ and using $\Gamma(n+1) = n!$, we obtain

$$C \int_1^{\infty} e^{-2\pi s}\, s^n\, \mathrm{d}s \leq C \int_0^{\infty} e^{-2\pi s}\, s^n\, \mathrm{d}s = C \int_0^{\infty} \frac{1}{(2\pi)^{n+1}} e^{-u}\, u^n\, \mathrm{d}u = \frac{C \cdot n!}{(2\pi)^n}.$$

Defining $C' := \frac{C \cdot n!}{(2\pi)^n}$ finishes the proof.

Lemma 8.9

There exists a constant $C > 0$ such that for all $r \geq 0$,

$$|I_1(r)|, |I_3(r)|, |I_5(r)| \leq C \int_1^\infty e^{-2\pi s}\, e^{-\pi r/s}\, \mathrm{d}s.$$

Proof for Lemma 8.9

We prove the bound only for $I_1(r)$, as the others are similar. With the change of variable $z = -1 + it$ for $t \in [0, 1]$, we have

$$I_1(r) = -i \int_0^1 \phi_0\left(\frac{-1}{it}\right) t^2 e^{-\pi i r} \cdot e^{-\pi r t}\, \mathrm{d}t.$$

With the change of variable $s = 1/t$, this becomes

$$I_1(r) = i \int_1^\infty \phi_0(is) \cdot s^{-4} \cdot e^{-\pi i r} \cdot e^{-\pi r/s}\, \mathrm{d}s,$$

so

$$|I_1(r)| \leq \int_1^\infty |\phi_0(is)| \cdot s^{-4} \cdot |e^{-\pi i r}| \cdot e^{-\pi r/s}\, \mathrm{d}s \leq \int_1^\infty |\phi_0(is)| \cdot e^{-\pi r/s}\, \mathrm{d}s.$$

By Corollary Corollary 8.5, we conclude that

$$|I_1(r)| \leq C_0 \int_1^\infty e^{-2\pi s}\, e^{-\pi r/s}\, \mathrm{d}s.$$

Lemma 8.10

There exist $C_1, C_2 > 0$ such that for all $r \geq 0$,

$$|I_2(r)|, |I_4(r)| \leq C_1 e^{-\pi r}$$

and

$$|I_6(r)| \leq C_2 \frac{e^{-\pi(r+2)}}{r+2}.$$

Proof for Lemma 8.10

For $I_2(r)$, parametrize z by $z = t + i$ for $t \in [-1, 0]$. Then

$$I_2(r) = \int_{-1}^0 \phi_0\left(\frac{-1}{t+1+i}\right) (t+1+i)^2 e^{\pi i r t} e^{-\pi r}\, \mathrm{d}t.$$

Since the function $z \mapsto \phi_0\left(\frac{-1}{z+1}\right)(z+1)^2$ is holomorphic and bounded on the contour, there exists $C > 0$ such that $|\phi_0\left(\frac{-1}{z+1}\right)(z+1)^2| \leq C$, and therefore

$$|I_2(r)| \leq \int_{-1}^{0} Ce^{-\pi r}\,\mathrm{d}t = Ce^{-\pi r}.$$

The bound for $I_4(r)$ is similar. /- source paragraph break -/ For $I_6(r)$, parametrize $z = it$ for $t \in [1, \infty)$, giving

$$I_6(r) = 2i\int_{1}^{\infty} \phi_0(it)e^{-\pi rt}\,\mathrm{d}t.$$

Using Corollary Corollary 8.5, the absolute value is bounded by

$$|I_6(r)| \leq 2\int_{1}^{\infty} |\phi_0(it)|e^{-\pi rt}\,\mathrm{d}t \leq \frac{2C_0}{\pi}\int_{1}^{\infty} e^{-2\pi t}e^{-\pi rt}\,\mathrm{d}t = \frac{2C_0}{\pi}\frac{e^{-\pi(r+2)}}{r+2}.$$

Combining these bounds, one can show that a is a Schwartz function.

Lemma 8.11

$a(x)$ is a Schwartz function.

Proof for Lemma 8.11

By Theorem Theorem 5.8, it suffices to show that the function is smooth and decays faster than any polynomial. Smoothness follows from differentiation under the integral, already formalized in `mathlib`. The decay follows from Lemmas Lemma 8.9 and Lemma 8.10, together with Lemma 8.8.

Lemma 8.12

$a(x)$ satisfies equation `eqn:a-fourier`.

Proof for Lemma 8.12

We recall that the Fourier transform of a Gaussian function is

$$\mathcal{F}(e^{\pi i\|x\|^2 z})(y) = z^{-4}\, e^{\pi i\|y\|^2\,(\frac{-1}{z})}.$$

Next, exchange contour integration in the z variable with Fourier transform in the x variable in the definition of a. This is justified because the corresponding double integral converges absolutely. In this way we obtain

$$\widehat{a}(y) = \int_{-1}^{i} \phi_0\Big(\frac{-1}{z+1}\Big)(z+1)^2\, z^{-4}\, e^{\pi i\|y\|^2\,(\frac{-1}{z})}\,dz + \int_{1}^{i} \phi_0\Big(\frac{-1}{z-1}\Big)(z-1)^2\, z^{-4}\, e^{\pi i\|y\|^2\,(\frac{-1}{z})}\,dz$$

$$-2\int_0^i \phi_0\Big(\frac{-1}{z}\Big)\, z^2\, z^{-4}\, e^{\pi i\|y\|^2\,(\frac{-1}{z})}\, dz + 2\int_i^{i\infty} \phi_0(z)\, z^{-4}\, e^{\pi i\|y\|^2\,(\frac{-1}{z})}\, dz.$$

Making the change of variables $w = \frac{-1}{z}$, we arrive at

$$\widehat{a}(y) = \int_1^i \phi_0\Big(1 - \frac{1}{w-1}\Big)\,(\frac{-1}{w}+1)^2\, w^2\, e^{\pi i\|y\|^2\, w}\, dw$$

$$+\int_{-1}^i \phi_0\Big(1 - \frac{1}{w+1}\Big)\,(\frac{-1}{w}-1)^2\, w^2\, e^{\pi i\|y\|^2\, w}\, dw$$

$$-2\int_{i\infty}^i \phi_0(w)\, e^{\pi i\|y\|^2\, w}\, dw + 2\int_i^0 \phi_0\Big(\frac{-1}{w}\Big)\, w^2\, e^{\pi i\|y\|^2\, w}\, dw.$$

Since ϕ_0 is 1-periodic we have

$$\begin{aligned}\widehat{a}(y) \;=& \int_1^i \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\, e^{\pi i\|y\|^2\, z}\, dz + \int_{-1}^i \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\, e^{\pi i\|y\|^2\, z}\, dz \\ &+2\int_i^{i\infty} \phi_0(z)\, e^{\pi i\|y\|^2\, z}\, dz - 2\int_0^i \phi_0\Big(\frac{-1}{z}\Big)\, z^2\, e^{\pi i\|y\|^2\, z}\, dz \\ &= a(y).\end{aligned}$$

This finishes the proof of the proposition.

Next, we check that a has double zeroes at all Λ_8-lattice points of length greater then $\sqrt{2}$. Using `eqn:phi0-bound`, `eqn:phi2-bound`, and `eqn:phi4-bound`, we can control the behavior of ϕ_0 near 0 and $i\infty$.

Corollary 8.13

We have

$$\phi_0\left(\frac{i}{t}\right) = O(e^{-2\pi/t}) \quad \text{as } t \to 0$$

$$\phi_0\left(\frac{i}{t}\right) = O(t^{-2}e^{2\pi t}) \quad \text{as } t \to \infty.$$

Proof for Corollary 8.13

The first estimate follows from `eqn:phi0-bound` with $z = i/t$. For the second estimate, by `eqn:phi0-trans-S`, `eqn:phi2-bound`, and `eqn:phi4-bound`,

$$\left|\phi_0\left(\frac{i}{t}\right)\right| \le |\phi_0(it)| + \frac{12}{\pi t}|\phi_{-2}(it)| + \frac{36}{\pi^2 t^2}|\phi_{-4}(it)|$$

$$\leq C_0 e^{-2\pi t} + \frac{12}{\pi t} \cdot C_{-2} + \frac{36}{\pi^2 t^2} \cdot C_{-4} e^{2\pi t} = O(t^{-2} e^{2\pi t}).$$

Lemma 8.14

For $r > \sqrt{2}$ we can express $a(r)$ in the form

$$a(r) = -4 \sin(\pi r^2/2)^2 \int_0^{i\infty} \phi_0\Big(\frac{-1}{z}\Big)\, z^2\, e^{\pi i r^2 z}\, dz.$$

Proof for Lemma 8.14

Denote the right-hand side by $d(r)$. Convergence of the integral for $r > \sqrt{2}$ follows from Corollary Corollary 8.13. We can write

$$d(r) = \int_{-1}^{i\infty-1} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\, e^{\pi i r^2 z}\, dz - 2\int_0^{i\infty} \phi_0\Big(\frac{-1}{z}\Big)\, z^2\, e^{\pi i r^2 z}\, dz$$
$$+ \int_1^{i\infty+1} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\, e^{\pi i r^2 z}\, dz.$$

From `eqn:phi0-trans-S` we deduce that if $r > \sqrt{2}$, then $\phi_0\Big(\frac{-1}{z}\Big)\, z^2\, e^{\pi i r^2 z} \to 0$ as $\Im(z) \to \infty$. Therefore we can deform the paths of integration and rewrite

$$d(r) = \int_{-1}^{i} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\, e^{\pi i r^2 z}\, dz + \int_i^{i\infty} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\, e^{\pi i r^2 z}\, dz$$
$$-2\int_0^{i} \phi_0\Big(\frac{-1}{z}\Big)\, z^2\, e^{\pi i r^2 z}\, dz - 2\int_i^{i\infty} \phi_0\Big(\frac{-1}{z}\Big)\, z^2\, e^{\pi i r^2 z}\, dz$$
$$+ \int_1^{i} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\, e^{\pi i r^2 z}\, dz + \int_i^{i\infty} \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\, e^{\pi i r^2 z}\, dz.$$

Now `eqn:phi0-trans-S` implies

$$\phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2 - 2\phi_0\Big(\frac{-1}{z}\Big)\, z^2 + \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2 = 2\phi_0(z),$$

and hence

$$d(r) = \int_{-1}^{i} \phi_0\Big(\frac{-1}{z+1}\Big)\,(z+1)^2\, e^{\pi i r^2 z}\, dz - 2\int_0^{i} \phi_0\Big(\frac{-1}{z}\Big)\, z^2\, e^{\pi i r^2 z}\, dz$$

$$+\int_1^i \phi_0\Big(\frac{-1}{z-1}\Big)\,(z-1)^2\,e^{\pi i r^2 z}\,dz + 2\int_i^{i\infty} \phi_0(z)\,e^{\pi i r^2 z}\,dz = a(r).$$

Finally, we find another convenient integral representation for a and compute values of $a(r)$ at $r = 0$ and $r = \sqrt{2}$.

Lemma 8.15

For $r \geq 0$ we have

$$a(r) = 4i\,\sin(\pi r^2/2)^2 \left(\frac{36}{\pi^3\,(r^2-2)} - \frac{8640}{\pi^3\,r^4} + \frac{18144}{\pi^3\,r^2} \right.$$

$$\left. + \int_0^\infty \left(t^2\,\phi_0\Big(\frac{i}{t}\Big) - \frac{36}{\pi^2}\,e^{2\pi t} + \frac{8640}{\pi}\,t - \frac{18144}{\pi^2} \right) e^{-\pi r^2 t}\,dt \right).$$

The integral converges absolutely for all $r \in \mathbb{R}_{\geq 0}$.

Proof for Lemma 8.15

Suppose that $r > \sqrt{2}$. Then by Proposition Lemma 8.14,

$$a(r) = 4i\,\sin(\pi r^2/2)^2 \int_0^\infty \phi_0(i/t)\,t^2\,e^{-\pi r^2 t}\,dt.$$

From `eqn:phi0-trans-S` we obtain the asymptotic expansion

$$\phi_0(i/t)\,t^2 = \frac{36}{\pi^2}\,e^{2\pi t} - \frac{8640}{\pi}\,t + \frac{18144}{\pi^2} + O(t^2\,e^{-2\pi t})$$

as $t \to \infty$. For $r > \sqrt{2}$,

$$\int_0^\infty \left(\frac{36}{\pi^2}\,e^{2\pi t} + \frac{8640}{\pi}\,t + \frac{18144}{\pi^2} \right) e^{-\pi r^2 t}\,dt = \frac{36}{\pi^3\,(r^2-2)} - \frac{8640}{\pi^3\,r^4} + \frac{18144}{\pi^3\,r^2}.$$

Therefore the claimed identity holds for $r > \sqrt{2}$. /- source paragraph break -/ On the other hand, the definition of a shows that $a(r)$ is analytic in a neighborhood of $[0, \infty)$, and the asymptotic expansion above implies that the right-hand side is also analytic there. Hence the identity holds on the whole interval $[0, \infty)$.

From `eqn:a-another-integral` we see that the values $a(r)$ lie in $i\mathbb{R}$ for all $r \in \mathbb{R}_{\geq 0}$.

Lemma 8.16

We have $a(0) = -\frac{i}{8640}$.

Proof for Lemma 8.16

These identities follow immediately from the previous proposition.

Now we construct function b. To this end we consider the function.

Definition 8.17

Define

$$h(z) := 128\frac{H_3(z) + H_4(z)}{H_2(z)^2}.$$

It is easy to see that $h \in M^!_{-2}(\Gamma_0(2))$. Indeed, first check that $h|_{-2}\gamma = h$ for all $\gamma \in \Gamma_0(2)$. Since $\Gamma_0(2)$ is generated by $\left(\begin{smallmatrix}1 & 0\\ 2 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$, it suffices to check invariance under their action, which follows immediately from the transformation laws of the theta-null forms and `eqn: h define`. Next analyze the poles of h. It is known that θ_{10} has no zeros in the upper half-plane, so h has poles only at the cusps. At the cusp $i\infty$, this modular form has Fourier expansion

$$h(z) = q^{-1} + 16 - 132q + 640q^2 - 2550q^3 + O(q^4).$$

Let $I = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$, and $S = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ be elements of Γ_1.

Definition 8.18

Define

$$\psi_I := h - h|_{-2}ST$$
$$\psi_T := \psi_I|_{-2}T$$
$$\psi_S := \psi_I|_{-2}S.$$

Lemma 8.19

$\psi_I(z)$, $\psi_S(z)$, and $\psi_T(z)$ can be written as

$$\psi_I(z) = \frac{H_4^3(5H_2^2 + 5H_2H_4 + 2H_4^2)}{2\Delta}$$
$$\psi_S(z) = -\frac{H_2^3(2H_2^2 + 5H_2H_4 + 5H_4^2)}{2\Delta}$$
$$\psi_T(z) = \frac{H_3^3(5H_2^2 - 5H_2H_3 + 2H_3^2)}{2\Delta}.$$

Proof for Lemma 8.19

By Lemma Lemma 7.33, we have

$$H_2|_{-2}ST = (-H_4)|_{-2}T = -H_3$$

$$H_3|_{-2}ST = (-H_3)|_{-2}T = -H_4$$

$$H_4|_{-2}ST = (-H_2)|_{-2}T = H_2.$$

Using these identities and Lemma Lemma 7.41, we can rewrite $\psi_I(z)$ as

$$\psi_I(z) = h(z) - h|_{-2}ST(z) = 128\frac{H_3 + H_4}{H_2^2} - 128\frac{-H_4 + H_2}{H_3^2}$$

$$= 128\frac{H_3^2(H_3 + H_4) - H_2^2(H_2 - H_4)}{H_2^2 H_3^2} = 128\frac{(H_2 + H_4)^2(H_2 + 2H_4) - H_2^2(H_2 - H_4)}{H_2^2 H_3^2}$$

$$= 128\frac{H_4(5H_2^2 + 5H_2H_4 + 2H_4^2)}{H_2^2 H_3^2} = \frac{H_4^3(5H_2^2 + 5H_2H_4 + 2H_4^2)}{2\Delta}.$$

Applying $|_{-2}S$ and $|_{-2}T$ to the expression for ψ_I proves the formulas for ψ_S and ψ_T.

Lemma 8.20

The Fourier expansions of these functions are

$$\psi_I(z) = q^{-1} + 144 + O(q^{1/2})$$

$$\psi_T(z) = q^{-1} + 144 + O(q^{1/2}).$$

Now we are ready to define b. Again, the definition here is slightly different from that in Viazovska (2017), where all contours are chosen to be straight lines.

Definition 8.21

Define $b_{\text{rad}} : \mathbb{R} \to \mathbb{C}$ by

$$b_{\text{rad}}(r) := J_1(r) + J_2(r) + J_3(r) + J_4(r) + J_5(r) + J_6(r)$$

where

$$J_1(r) := \int_{-1}^{-1+i} \psi_T(z) e^{\pi i r z} \, \mathrm{d}z$$

$$J_2(r) := \int_{-1+i}^{i} \psi_T(z) e^{\pi i r z} \, \mathrm{d}z$$

$$J_3(r) := \int_{1}^{1+i} \psi_T(z) e^{\pi i r z} \, \mathrm{d}z$$

$$J_4(r) := \int_{1+i}^{i} \psi_T(z) e^{\pi i r z} \, \mathrm{d}z$$

$$J_5(r) := -2\int_0^i \psi_I(z)e^{\pi i r z}\,\mathrm{d}z$$
$$J_6(r) := -2\int_i^{i\infty} \psi_S(z)e^{\pi i r z}\,\mathrm{d}z.$$

Here all contours are straight line segments. Define $b : \mathbb{R}^8 \to \mathbb{C}$ by $b(x) := b_{\mathrm{rad}}(\|x\|^2)$.

Now we prove that b is a Schwartz function and satisfies `eqn:b-fourier`. As in the case of $a(x)$, we start with an exponential bound for $\psi_S(z)$ and $\psi_I(z)$.

Lemma 8.22

There exist constants $C_I, C_S, C_T > 0$ such that

$$|\psi_I(z)| \le C_I e^{2\pi \Im z}$$
$$|\psi_T(z)| \le C_T e^{2\pi \Im z}$$
$$|\psi_S(z)| \le C_S e^{-\pi \Im z}.$$

Proof for Lemma 8.22

The proof is similar to that of Lemma Corollary 8.5 and follows from Lemma Lemma 8.4 together with the fact that the vanishing orders of the numerators of ψ_I, ψ_T, and ψ_S at infinity are respectively 0, 0, and $\frac{3}{2}$.

Lemma 8.23

There exists a constant $C > 0$ such that

$$|J_1(r)|, |J_3(r)|, |J_5(r)| \le C\int_1^\infty e^{-\pi s}e^{\pi r/s}\,\mathrm{d}s.$$

Lemma 8.24

There exist $C_1, C_2 > 0$ such that

$$|J_2(r)|, |J_4(r)| \le C_1 e^{-\pi r}$$
$$|J_6(r)| \le C_2 \frac{e^{\pi(r+1)}}{r+1}.$$

Lemma 8.25

$b(x)$ is a Schwartz function.

Proof for Lemma 8.25

Similar to the proof of Lemma 8.11.

Lemma 8.26

$b(x)$ satisfies equation `eqn:b-fourier`.

Proof for Lemma 8.26

We repeat the argument used in the proof of Proposition Lemma 8.12. Using the Gaussian Fourier identity and exchanging the contour integration in z with the Fourier transform in x, we get

$$\mathcal{F}(b)(x) = \int_{-1}^{i} \psi_T(z)\, z^{-4}\, e^{\pi i \|x\|^2 (\frac{-1}{z})}\, dz + \int_{1}^{i} \psi_T(z)\, z^{-4}\, e^{\pi i \|x\|^2 (\frac{-1}{z})}\, dz$$

$$-2 \int_{0}^{i} \psi_I(z)\, z^{-4}\, e^{\pi i \|x\|^2 (\frac{-1}{z})}\, dz - 2 \int_{i}^{i\infty} \psi_S(z)\, z^{-4}\, e^{\pi i \|x\|^2 (\frac{-1}{z})}\, dz.$$

With the change of variables $w = \frac{-1}{z}$, we arrive at

$$\mathcal{F}(b)(x) = \int_{1}^{i} \psi_T\Big(\frac{-1}{w}\Big)\, w^2\, e^{\pi i \|x\|^2 w}\, dw + \int_{-1}^{i} \psi_T\Big(\frac{-1}{w}\Big)\, w^2\, e^{\pi i \|x\|^2 w}\, dw$$

$$-2 \int_{i\infty}^{i} \psi_I\Big(\frac{-1}{w}\Big)\, w^2\, e^{\pi i \|x\|^2 w}\, dw - 2 \int_{i}^{0} \psi_S\Big(\frac{-1}{w}\Big)\, w^2\, e^{\pi i \|x\|^2 w}\, dw.$$

Now the definitions imply

$$\psi_T|_{-2}S = -\psi_T$$

$$\psi_I|_{-2}S = \psi_S$$

$$\psi_S|_{-2}S = \psi_I,$$

so this becomes

$$\mathcal{F}(b)(x) = \int_{1}^{i} -\psi_T(z)\, e^{\pi i \|x\|^2 z}\, dz + \int_{-1}^{i} -\psi_T(z)\, e^{\pi i \|x\|^2 z}\, dz$$

$$+2 \int_{i}^{i\infty} \psi_S(z)\, e^{\pi i \|x\|^2 z}\, dz + 2 \int_{0}^{i} \psi_I(z)\, e^{\pi i \|x\|^2 z}\, dw.$$

From the definition of b, we conclude that

$$\mathcal{F}(b)(x) = -b(x).$$

Now we regard the radial function b as a function on $\mathbb{R}_{\geq 0}$ and check that it has double roots at the Λ_8 points.

Corollary 8.27

We have

$$\psi_I(it) = O(t^2 e^{\pi/t}) \quad \text{as } t \to 0$$
$$\psi_I(it) = O(e^{2\pi t}) \quad \text{as } t \to \infty.$$

Proof for Corollary 8.27

By `eqn:psiS-define`,

$$\psi_I(it) = (it)^{-2}\psi_S\left(\frac{-1}{it}\right) = -t^{-2}\psi_S\left(\frac{i}{t}\right),$$

and combined with `eqn:psiS-bound` this gives `eqn:psiI-near-0`. Equation `eqn:psiI-near-infty` follows from Lemma 8.22.

Lemma 8.28

For $r > \sqrt{2}$, the function $b(r)$ can be expressed as

$$b(r) = -4\sin(\pi r^2/2)^2 \int_0^{i\infty} \psi_I(z)\, e^{\pi i r^2 z}\, dz.$$

Proof for Lemma 8.28

Denote the right-hand side by $c(r)$. By Corollary Corollary 8.27, the integral converges for $r > \sqrt{2}$. Rewrite it as

$$c(r) = \int_{-1}^{i\infty-1} \psi_I(z+1)\, e^{\pi i r^2 z}\, dz - 2\int_0^{i\infty} \psi_I(z)\, e^{\pi i r^2 z}\, dz + \int_1^{i\infty+1} \psi_I(z-1)\, e^{\pi i r^2 z}\, dz.$$

From the Fourier expansion of ψ_I, we know that $\psi_I(z) = e^{-2\pi i z} + O(1)$ as $\Im(z) \to \infty$. Because $r^2 > 2$, we can deform the paths of integration and write

$$\int_{-1}^{i\infty-1} \psi_I(z+1)\, e^{\pi i r^2 z}\, dz = \int_{-1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz + \int_i^{i\infty} \psi_T(z)\, e^{\pi i r^2 z}\, dz$$

$$\int_1^{i\infty+1} \psi_I(z-1)\, e^{\pi i r^2 z}\, dz = \int_{-1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz + \int_i^{i\infty} \psi_T(z)\, e^{\pi i r^2 z}\, dz.$$

Hence

$$c(r) = \int_{-1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz + \int_{1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz - 2\int_{0}^{i} \psi_I(z)\, e^{\pi i r^2 z}\, dz$$

$$+2\int_{i}^{i\infty} (\psi_T(z) - \psi_I(z))\, e^{\pi i r^2 z}\, dz.$$

Next, the functions ψ_I, ψ_T, and ψ_S satisfy

$$\psi_T + \psi_S = \psi_I.$$

Using this identity, we find

$$c(r) = \int_{-1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz + \int_{1}^{i} \psi_T(z)\, e^{\pi i r^2 z}\, dz - 2\int_{0}^{i} \psi_I(z)\, e^{\pi i r^2 z}\, dz$$

$$-2\int_{i}^{i\infty} \psi_S(z)\, e^{\pi i r^2 z}\, dz = b(r).$$

At the end of this section we find another integral representation of $b(r)$ for $r \in \mathbb{R}_{\geq 0}$ and compute special values of b.

Lemma 8.29

For $r \geq 0$ we have

$$b(r) = 4i\, \sin(\pi r^2/2)^2 \left(\frac{144}{\pi\, r^2} + \frac{1}{\pi\,(r^2-2)} + \int_0^{\infty} \left(\psi_I(it) - 144 - e^{2\pi t}\right)\, e^{-\pi r^2 t}\, dt \right).$$

The integral converges absolutely for all $r \in \mathbb{R}_{\geq 0}$.

Proof for Lemma 8.29

The proof is analogous to the proof of Proposition Lemma 8.15. First suppose that $r > \sqrt{2}$. Then by Proposition Lemma 8.28,

$$b(r) = 4i\, \sin(\pi r^2/2)^2 \int_0^{\infty} \psi_I(it)\, e^{-\pi r^2 t}\, dt.$$

From the Fourier expansion of ψ_I we obtain

$$\psi_I(it) = e^{2\pi t} + 144 + O(e^{-\pi t})$$

as $t \to \infty$. For $r > \sqrt{2}$,

$$\int_0^\infty \left(e^{2\pi t} + 144\right)\, e^{-\pi r^2 t}\, dt = \frac{1}{\pi\,(r^2 - 2)} + \frac{144}{\pi\, r^2}.$$

Therefore the identity holds for $r > \sqrt{2}$. /- source paragraph break -/ On the other hand, the definition of b shows that $b(r)$ is analytic in a neighborhood of $[0, \infty)$, and the asymptotic expansion above implies that the right-hand side is also analytic there. Hence the identity holds on the whole interval $[0, \infty)$.

From `eqn:b-another-integral` we see that $b(r) \in i\mathbb{R}$ for all $r \in \mathbb{R}_{\geq 0}$.

Lemma 8.30

We have $b(0) = 0$.

Proof for Lemma 8.30

These identities follow immediately from the previous proposition.

9 Proof of the Optimal Function Inequalities

Lemma 9.1

Consider the function $A : (0, \infty) \to \mathbb{C}$ defined as

$$A(t) := -t^2\phi_0(i/t) - \frac{36}{\pi^2}\,\psi_I(it).$$

Then

$$A(t) < 0$$

for all $t > 0$.

Lemma 9.2

Consider the function $B : (0, \infty) \to \mathbb{C}$ defined as

$$B(t) := -t^2\phi_0(i/t) + \frac{36}{\pi^2}\,\psi_I(it).$$

Then

$$B(t) > 0$$

for all $t > 0$.

Here we formalize the proof of the inequalities by Lee (2024). First, we can rewrite the inequality in Lemma 9.1 as follows.

Definition 9.3

Define two quasi-modular forms by

$$F(z) = (E_2(z)E_4(z) - E_6(z))^2$$

$$G(z) = H_2(z)^3(2H_2(z)^2 + 5H_2(z)H_4(z) + 5H_4(z)^2).$$

Lemma 9.4

We have

$$\phi_0 = \frac{F}{\Delta}$$
$$\psi_S = -\frac{1}{2}\frac{G}{\Delta}.$$

Proof for Lemma 9.4

Equation `eqn:phi0-F` is clear from the definitions. Equation `eqn:psiS-G` is already shown in Lemma Lemma 8.19.

Lemma 9.5

Inequality `eqn:ineqA` and `eqn:ineqB` are equivalent to

$$F(it) + \frac{18}{\pi^2}G(it) > 0$$

$$F(it) - \frac{18}{\pi^2}G(it) < 0$$

respectively.

Proof for Lemma 9.5

By `eqn:psiS-define`,

$$\psi_I(it) = (\psi_S|_{-2}S)(it) = (it)^2\psi_S\left(-\frac{1}{it}\right) = -t^2\psi_S\left(\frac{i}{t}\right).$$

Combined with Lemma Lemma 9.4 we can rewrite `eqn:ineqA` as

$$A(t) = -t^2\phi_0\left(\frac{i}{t}\right) + \frac{36}{\pi^2}\psi_S\left(\frac{i}{t}\right) < 0 \Leftrightarrow \frac{F(it)}{\Delta(it)} + \frac{18}{\pi^2}\frac{G(it)}{\Delta(it)} > 0$$

for $t > 0$, which is equivalent to `eqn:ineqAnew` by Corollary Corollary 7.25. Equivalences of `eqn:ineqB` and `eqn:ineqBnew` follows similarly; just change the sign.

Now, the first inequality `eqn:ineqAnew` follows from the positivity of each $F(it)$ and $G(it)$.

Lemma 9.6

For all $t > 0$, we have $F(it) > 0$ and $G(it) > 0$.

Proof for Lemma 9.6

By Ramanujan's identity eqn:DE4, we have $F(z) = 9E_4'(z)^2$, and hence

$$F(it) = 9E_4'(it)^2 = 9\left(240\sum_{n\geq 1} n\sigma_3(n)e^{-2\pi nt}\right)^2 > 0.$$

The inequality $G(it) > 0$ follows from positivity of $H_2(it)$ and $H_4(it)$ in Corollary 7.42.

Corollary 9.7

Equation eqn:ineqAnew holds.

Proof for Corollary 9.7

This directly follows from Lemma Lemma 9.6.

To prove the second inequality, we need some identities satisfied by F and G.

Lemma 9.8

F and G satisfy the following differential equations:

$$\partial_{12}\partial_{10}F - \frac{5}{6}E_4F = 7200\Delta(-E_2')$$

$$\partial_{12}\partial_{10}G - \frac{5}{6}E_4G = -640\Delta H_2.$$

Proof for Lemma 9.8

Both identities can be shown by direct computations. By Ramanujan's identities (Theorem Theorem 7.48) and the product rule for Serre derivatives (Theorem Theorem 7.51), we have

$$\partial_5(E_2E_4 - E_6) = (E_2E_4 - E_6)' - \frac{5}{12}E_2(E_2E_4 - E_6)$$

$$= \frac{E_2^2 - E_4}{12}\cdot E_4 + E_2\cdot\frac{E_2E_4 - E_6}{3} - \frac{E_2E_6 - E_4^2}{2} - \frac{5}{12}E_2(E_2E_4 - E_6) = -\frac{5}{12}(E_2E_6 - E_4^2)$$

and

$$\partial_7(E_2E_6 - E_4^2) = (E_2E_6 - E_4^2)' - \frac{7}{12}E_2(E_2E_6 - E_4^2)$$

$$= \frac{E_2^2 - E_4}{12}\cdot E_6 + E_2\cdot\frac{E_2E_6 - E_4^2}{2} - 2E_4\cdot\frac{E_2E_4 - E_6}{3} - \frac{7}{12}E_2(E_2E_6 - E_4^2) = -\frac{7}{12}E_4(E_2E_4 - E_6).$$

Using these, we compute

$$\partial_{10}F = \partial_{10}(E_2E_4-E_6)^2 = 2(E_2E_4-E_6)\partial_5(E_2E_4-E_6) = -\frac{5}{6}(E_2E_4-E_6)(E_2E_6-E_4^2),$$

and then

$$\partial_{12}\partial_{10}F = -\frac{5}{6}\partial_{12}((E_2E_4 - E_6)(E_2E_6 - E_4^2))$$

$$= -\frac{5}{6}(\partial_5(E_2E_4-E_6))(E_2E_6-E_4^2)-\frac{5}{6}(E_2E_4-E_6)(\partial_7(E_2E_6-E_4^2)) = \frac{25}{72}(E_2E_6-E_4^2)^2+\frac{35}{72}E_4(E_2E_4-E_6)^2.$$

Hence

$$\partial_{12}\partial_{10}F-\frac{5}{6}E_4F = \frac{25}{72}((E_2E_6-E_4^2)^2-E_4(E_2E_4-E_6)^2) = \frac{25}{72}(-E_2^2E_4^3+E_2^2E_6^2+E_4^4-E_4E_6^3) = -\frac{25}{72}(E_4^3-E_6^2)$$

This proves equation `eqn:ddf`. The second is proved similarly, using Proposition Lemma 7.50 and Lemma Lemma 7.41.

Corollary 9.9

Equation `eqn:ddf` is positive and equation `eqn:ddg` is negative on the positive imaginary axis.

Proof for Corollary 9.9

From `eqn:E2` and Lemma Corollary 7.25,

$$7200(-E_2'(it))\Delta(it) = 7200 \cdot 24\left(\sum_{n\geq 1} n\sigma_1(n)e^{-2\pi nt}\right)\cdot\Delta(it) > 0.$$

Negativity of equation `eqn:ddg`, namely $-640\Delta(it)H_2(it) < 0$, follows from Corollary 7.42 and Corollary 7.25.

The second inequality follows from the following two observations. Since $G(it) > 0$ for all $t > 0$, we can define the quotient

$$Q(t) := \frac{F(it)}{G(it)}$$

as a function on $(0, \infty)$.

Lemma 9.10

We have

$$\lim_{t\to 0^+} Q(t) = \frac{18}{\pi^2}.$$

Proof for Lemma 9.10

We have

$$\lim_{t\to 0^+} Q(t) = \lim_{t\to 0^+} \frac{F(it)}{G(it)} = \lim_{t\to\infty} \frac{F(i/t)}{G(i/t)}.$$

By using the transformation laws of Eisenstein series `eqn:E2-S-transform`, `eqn:Ek-trans-S` (for $k = 4, 6$) and the thetanull functions, `eqn:H2-transform-S`, `eqn:H4-transform-S`, we get

$$F\left(\frac{i}{t}\right) = t^{12}F(it) - \frac{12t^{11}}{\pi}(E_2(it)E_4(it) - E_6(it))E_4(it) + \frac{36t^{10}}{\pi^2}E_4(it)^2$$

and

$$G\left(\frac{i}{t}\right) = t^{10}H_4(it)^3(2H_4(it)^2 + 5H_4(it)H_2(it) + 5H_2(it)^2).$$

Since F, $E_2E_4 - E_6$ and H_2 are cusp forms, we have $\lim_{t\to\infty} t^k A(it) = 0$ when $A(z)$ is one of these forms and $k \geq 0$. From $\lim_{t\to\infty} E_4(it) = 1 = \lim_{t\to\infty} H_4(it)$, we get

$$\begin{aligned}\lim_{t\to\infty} \frac{F(i/t)}{G(i/t)} &= \lim_{t\to\infty} \frac{t^{12}F(it) - \frac{12t^{11}}{\pi}(E_2(it)E_4(it) - E_6(it))E_4(it) + \frac{36t^{10}}{\pi^2}E_4(it)^2}{t^{10}H_4(it)^3(2H_4(it)^2 + 5H_4(it)H_2(it) + 5H_2(it)^2)} \\ &= \lim_{t\to\infty} \frac{t^2F(it) - \frac{12t}{\pi}(E_2(it)E_4(it) - E_6(it))E_4(it) + \frac{36}{\pi^2}E_4(it)^2}{H_4(it)^3(2H_4(it)^2 + 5H_4(it)H_2(it) + 5H_2(it)^2)} \\ &= \frac{18}{\pi^2}.\end{aligned}$$

Lemma 9.11

Let F be a quasimodular form whose vanishing order at infinity is $n_0 > 0$, that is, $F(z) = \sum_{n\geq n_0} a_n q^n$ with $a_{n_0} \neq 0$. Then

$$\lim_{t\to\infty} \frac{F'(it)}{F(it)} = n_0.$$

Proof for Lemma 9.11

By Lemma Lemma 7.44,

$$\lim_{t\to\infty} \frac{F'(it)}{F(it)} = \lim_{t\to\infty} \frac{\sum_{n\geq n_0} n a_n e^{-2\pi nt}}{\sum_{n\geq n_0} a_n e^{-2\pi nt}} = \lim_{t\to\infty} \frac{n_0 a_{n_0} e^{-2\pi n_0 t} + O(e^{-2\pi(n_0+1)t})}{a_{n_0} e^{-2\pi n_0 t} + O(e^{-2\pi(n_0+1)t})} = n_0.$$

Lemma 9.12

The function $t \mapsto Q(t)$ is strictly decreasing.

Proof for Lemma 9.12

It is enough to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{F(it)}{G(it)}\right) < 0 \Leftrightarrow (-2\pi)\frac{F'(it)G(it) - F(it)G'(it)}{G(it)^2} < 0 \Leftrightarrow F'(it)G(it) - F(it)G'(it) > 0 \Leftrightarrow (\partial_{10}F)(it)G($$

Let $\mathcal{L}_{1,0} := (\partial_{10}F)G - F(\partial_{10}G) = F'G - FG'$. Then

$$\frac{\mathcal{L}_{1,0}}{FG} = \frac{F'G - FG'}{FG} = \frac{F'}{F} - \frac{G'}{G}.$$

The vanishing orders of F and G at infinity are 2 and $\frac{3}{2}$ respectively, so by Lemma 9.11 we have

$$\lim_{t\to\infty} \frac{\mathcal{L}_{1,0}(it)}{F(it)G(it)} = \lim_{t\to\infty}\left(\frac{F'(it)}{F(it)} - \frac{G'(it)}{G(it)}\right) = 2 - \frac{3}{2} = \frac{1}{2} > 0,$$

so $\mathcal{L}_{1,0}(it) > 0$ for sufficiently large t. Its Serre derivative is positive by Corollary 9.9:

$$\partial_{22}\mathcal{L}_{1,0} = (\partial_{12}\partial_{10}F)G - F(\partial_{12}\partial_{10}G) = \Delta(7200(-E_2')G + 640H_2F) > 0.$$

Hence $\mathcal{L}_{1,0}(it) > 0$ by Theorem 7.52, and the monotonicity follows.

Corollary 9.13

Equation `eqn:ineqBnew` holds.

Proof for Corollary 9.13

We have

$$\frac{F(it)}{G(it)} = Q(t) < \lim_{u\to 0^+} Q(u) = \frac{18}{\pi^2},$$

and by Lemma Lemma 9.6 the desired inequality follows.

Theorem 9.14

The function

$$g(x) := \frac{\pi\, i}{8640}a(x) + \frac{i}{240\pi}\, b(x)$$

satisfies conditions `eqn:g1`–`eqn:g3`.

Proof for Theorem 9.14

First, we prove the first sign condition. By Propositions Lemma 8.14 and Lemma 8.28 we know that for $r > \sqrt{2}$,

$$g(r) = \frac{\pi}{2160}\, \sin(\pi r^2/2)^2 \int_0^\infty A(t)\, e^{-\pi r^2 t}\, dt$$

where

$$A(t) = -t^2\phi_0(i/t) - \frac{36}{\pi^2}\,\psi_I(it).$$

From Proposition Lemma 9.1 we know that $A(t) < 0$ for $t \in (0, \infty)$, so this identity implies the first sign condition. /- source paragraph break -/ Next, we prove the Fourier-side sign condition. By Propositions Lemma 8.15 and Lemma 8.29 we know that for $r > 0$,

$$\widehat{g}(r) = \frac{\pi}{2160}\,\sin(\pi r^2/2)^2 \int_0^\infty B(t)\, e^{-\pi r^2 t}\, dt$$

where

$$B(t) = -t^2\phi_0(i/t) + \frac{36}{\pi^2}\,\psi_I(it).$$

/- source paragraph break -/ Finally, the normalization $g(0) = \widehat{g}(0) = 1$ follows readily from Propositions Lemma 8.16 and Lemma 8.30. This finishes the proof of Theorems Theorem 9.14 and Theorem 6.3.

Dependency Graph

The dependency graph is available in the HTML output.

Blueprint Summary

The Blueprint summary is available in the HTML output.

Blueprint Bibliography

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[1] `https://arxiv.org/abs/2406.14659`