

Noperthedron

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Contents

Contents	i
1 Introduction	1
2 The Noperthedron	2
2.1 Definition of the Noperthedron	2
2.2 Refined Rupert's property for the Noperthedron	3
3 Bounding Rotations	4
4 Preliminaries	8
4.1 Rupert Sets	8
4.2 Poses	8
4.3 Pointsymmetry and Rupertness	8
5 The Global Theorem	10
6 The Local Theorem	13
7 Rational Versions	17
8 Computational Step	23
9 Main Theorems	25
Dependency Graph	27
Blueprint Summary	28
Blueprint Bibliography	29

1 Introduction

We follow for the most part the structure of Steininger and Yurkevich (2025).

2 The Noperthedron

2.1 Definition of the Noperthedron

We define three points $C_1, C_2, C_3 \in \mathbb{Q}^3$.

$$C_1 := \frac{1}{259375205} \begin{pmatrix} 152024884 \\ 0 \\ 210152163 \end{pmatrix}, \quad C_2 := \frac{1}{10^{10}} \begin{pmatrix} 6632738028 \\ 6106948881 \\ 3980949609 \end{pmatrix},$$

$$C_3 := \frac{1}{10^{10}} \begin{pmatrix} 8193990033 \\ 5298215096 \\ 1230614493 \end{pmatrix}.$$

Lemma 2.1.1

$$\|C_1\| = 1, \frac{98}{100} < \|C_2\| < \frac{99}{100}, \text{ and } \frac{98}{100} < \|C_3\| < \frac{99}{100}.$$

Proof for Lemma 2.1.1

Trivial arithmetic.

Rotations about the x, y, z axes $R_x, R_y, R_z : \mathbb{R} \rightarrow \mathbb{R}^{3 \times 3}$ are defined in the usual way:

$$R_x(\alpha) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix}, \quad R_y(\alpha) := \begin{pmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{pmatrix},$$

$$R_z(\alpha) := \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We define a 30-element set $\mathcal{C}_{30}$

$$\mathcal{C}_{30} := \left\{ (-1)^\ell R_z \left(\frac{2\pi k}{15} \right) : k = 0, \dots, 14; \ell = 0, 1 \right\}.$$

of rotations.

We write $\mathcal{C}_{30} \cdot P = \{cP \text{ for } c \in \mathcal{C}_{30}\}$ for the orbit of P under the action of $\mathcal{C}_{30}$.

Definition 2.1.2

The Noperthedron is the polyhedron given by the vertex set

$$\mathcal{C}_{30} \cdot C_1 \cup \mathcal{C}_{30} \cdot C_2 \cup \mathcal{C}_{30} \cdot C_3.$$

Lemma 2.1.3

The norm of any vertex in the Noperthedron is no more than 1.

Proof for Lemma 2.1.3

Evident from definitions.

Definition 2.1.4

A set $S \subseteq \mathbb{R}^3$ is *point-symmetric* if $x \in S$ implies $-x \in S$.

Lemma 2.1.5

The noperthedron is point-symmetric.

Proof for Lemma 2.1.5

Follows directly from the definition of the exact vertex set.

2.2 Refined Rupert's property for the Noperthedron

Lemma 2.2.1

Let $\mathbf{P} = \mathbf{NOP}$, then for all $\theta, \varphi, \alpha \in \mathbb{R}$, the following three identities hold (as sets):

$$\begin{aligned} M(\theta + 2\pi/15, \varphi) \cdot \mathbf{P} &= M(\theta, \varphi) \cdot \mathbf{P}, \\ R(\alpha + \pi)M(\theta, \varphi) \cdot \mathbf{P} &= R(\alpha)M(\theta, \varphi) \cdot \mathbf{P}, \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M(\theta, \varphi) \cdot \mathbf{P} &= M(\theta + \pi/15, \pi - \varphi) \cdot \mathbf{P}. \end{aligned}$$

Proof for Lemma 2.2.1

See polyhedron.without.rupert, Lemma 7.

Corollary 2.2.2

If the noperthedron is Rupert, then there exists a solution with

$$\begin{aligned} \theta_1, \theta_2 &\in [0, 2\pi/15] \subset [0, 0.42], \\ \varphi_1 &\in [0, \pi] \subset [0, 3.15], \\ \varphi_2 &\in [0, \pi/2] \subset [0, 1.58], \\ \alpha &\in [-\pi/2, \pi/2] \subset [-1.58, 1.58]. \end{aligned}$$

Proof for Corollary 2.2.2

See polyhedron.without.rupert, Lemma 8.

3 Bounding Rotations

Lemma 3.1

For any $\alpha, \theta, \varphi \in \mathbb{R}$ and $a \in \{x, y, z\}$ one has $\|R(\alpha)\| = \|R_a(\alpha)\| = \|M(\theta, \phi)\| = 1$.

Proof for Lemma 3.1

See polyhedron.without.rupert, Lemma 9.

Lean code for «code:lem:RaAlpha»

```
theorem bp_Rx_norm_one (α : ℝ) : ||RxL α|| = 1 := by
  simp using Bounding.Rx_norm_one α

theorem bp_Ry_norm_one (α : ℝ) : ||RyL α|| = 1 := by
  simp using Bounding.Ry_norm_one α

theorem bp_Rz_norm_one (α : ℝ) : ||RzL α|| = 1 := by
  simp using Bounding.Rz_norm_one α

theorem bp_rotR_norm_one (α : ℝ) : ||rotR α|| = 1 := by
  simp using Bounding.rotR_norm_one α

theorem bp_rotM_norm_one (θ φ : ℝ) : ||rotM θ φ|| = 1 := by
  simp using Bounding.rotM_norm_one θ φ
```

Lemma 3.2

Let $\epsilon > 0$, $|\alpha - \bar{\alpha}| \leq \epsilon$ and $a \in \{x, y, z\}$ then $\|R_a(\alpha) - R_a(\bar{\alpha})\| = \|R(\alpha) - R(\bar{\alpha})\| < \epsilon$.

Proof for Lemma 3.2

See polyhedron.without.rupert, Lemma 10.

Lean code for «code:lem:RaRa»

```

theorem bp_norm_rotR_sub_rotR_lt { $\varepsilon$   $\alpha$   $\alpha_$  :  $\mathbb{R}$ } ( $h\varepsilon$  :  $0 < \varepsilon$ ) ( $h\alpha$ 
 $\hookrightarrow$  :  $|\alpha - \alpha_| \leq \varepsilon$ ) :
   $\|\text{rotR } \alpha - \text{rotR } \alpha_ \| < \varepsilon$  := by
  simp using Bounding.norm_rotR_sub_rotR_lt  $h\varepsilon$   $h\alpha$ 

theorem bp_norm_RxL_sub_RxL_eq { $\alpha$   $\alpha_$  :  $\mathbb{R}$ } :
   $\|\text{RxL } \alpha - \text{RxL } \alpha_ \| = \|\text{rotR } \alpha - \text{rotR } \alpha_ \|$  := by
  simp using Bounding.norm_RxL_sub_RxL_eq ( $\alpha := \alpha$ ) ( $\alpha_ := \alpha_$ )

theorem bp_norm_RyL_sub_RyL_eq { $\alpha$   $\alpha_$  :  $\mathbb{R}$ } :
   $\|\text{RyL } \alpha - \text{RyL } \alpha_ \| = \|\text{rotR } \alpha - \text{rotR } \alpha_ \|$  := by
  simp using Bounding.norm_RyL_sub_RyL_eq ( $\alpha := \alpha$ ) ( $\alpha_ := \alpha_$ )

theorem bp_norm_RzL_sub_RzL_eq { $\alpha$   $\alpha_$  :  $\mathbb{R}$ } :
   $\|\text{RzL } \alpha - \text{RzL } \alpha_ \| = \|\text{rotR } \alpha - \text{rotR } \alpha_ \|$  := by
  simp using Bounding.norm_RzL_sub_RzL_eq ( $\alpha := \alpha$ ) ( $\alpha_ := \alpha_$ )

```

Lemma 3.3

For any $\alpha, \beta \in \mathbb{R}$ one has $\|R_x(\alpha)R_y(\beta) - \text{id}\| \leq \sqrt{\alpha^2 + \beta^2}$ with strict inequality unless $\alpha = \beta = 0$.

Proof for Lemma 3.3

The composition $R_d(\alpha)R_{d'}(\beta)$ is a rotation, i.e. conjugate to some $R_z(\gamma)$ by an isometry, which gives $\|R_d(\alpha)R_{d'}(\beta) - \text{id}\|^2 = 2(1 - \cos \gamma) = 3 - \text{tr}(R_d(\alpha)R_{d'}(\beta)) = 3 - (\cos \alpha + \cos \beta + \cos \alpha \cos \beta)$. Writing the right hand side as $2(1 - \cos \alpha) + 2(1 - \cos \beta) - (1 - \cos \alpha)(1 - \cos \beta)$ and using $2(1 - \cos x) \leq x^2$ (with equality only at $x = 0$) yields the bound and the equality condition. This is a direct route to polyhedron.without.rupert, Lemma 12, that avoids the Jensen-inequality argument of Lemma 11 and the doubling induction.

Lean code for «code:lem:RxRy»

```

theorem bp_lemma12 { $d$   $d'$  : Fin 3} { $\alpha$   $\beta$  :  $\mathbb{R}$ } ( $hd$  :  $d \neq d'$ ) :
   $\|\text{rot3 } d \alpha \circ \text{L rot3 } d' \beta - 1\| \leq \sqrt{\alpha^2 + \beta^2}$  := by
  simp using Bounding.lemma12 ( $d := d$ ) ( $d' := d'$ ) ( $\alpha := \alpha$ ) ( $\beta$ 
 $\hookrightarrow$  :=  $\beta$ )  $hd$ 

theorem bp_lemma12_lt_of_ne { $d$   $d'$  : Fin 3} { $\alpha$   $\beta$  :  $\mathbb{R}$ } ( $hd$  :  $d \neq$ 
 $d'$ )
  ( $h$  :  $\neg (\alpha = 0 \wedge \beta = 0)$ ) :
   $\|\text{rot3 } d \alpha \circ \text{L rot3 } d' \beta - 1\| < \sqrt{\alpha^2 + \beta^2}$  := by
  simp using Bounding.lemma12_lt_of_ne  $hd$   $h$ 

```

Lemma 3.4

Let $\epsilon > 0$ and $|\theta - \bar{\theta}|, |\varphi - \bar{\varphi}| \leq \epsilon$ then $\|M(\theta, \phi) - M(\bar{\theta}, \bar{\varphi})\|, \|X(\theta, \varphi) - X(\bar{\theta}, \bar{\varphi})\| < \sqrt{2}\epsilon$.

Proof for Lemma 3.4

See polyhedron.without.rupert, Lemma 13.

Lean code for «code:lem:sqrt2»

```
theorem bp_norm_M_sub_lt {ε θ θ_ φ φ_ : ℝ} (hε : 0 < ε) (hθ :
  ⇐ |θ - θ_| ≤ ε) (hφ : |φ - φ_| ≤ ε) :
  ||rotM θ φ - rotM θ_ φ_|| < √2 * ε := by
  simp using Bounding.norm_M_sub_lt hε hθ hφ

theorem bp_norm_X_sub_lt {ε θ θ_ φ φ_ : ℝ} (hε : 0 < ε) (hθ :
  ⇐ |θ - θ_| ≤ ε) (hφ : |φ - φ_| ≤ ε) :
  ||vecX θ φ - vecX θ_ φ_|| < √2 * ε := by
  simp using Bounding.norm_X_sub_lt hε hθ hφ
```

Lemma 3.5

Let $P \in \mathbb{R}^3$ with $\|P\| \leq 1$. Further, let $\epsilon > 0$ and $\bar{\theta}, \bar{\varphi}, \theta, \phi \in \mathbb{R}$ such that $|\bar{\theta} - \theta|, |\bar{\varphi} - \phi| \leq \epsilon$. If $\langle X(\bar{\theta}, \bar{\varphi}), P \rangle > \sqrt{2}\epsilon$ then $\langle X(\theta, \phi), P \rangle > 0$.

Proof for Lemma 3.5

See polyhedron.without.rupert, Lemma 14.

Lean code for «code:lem:XPgt0»

```
theorem bp_XPgt0 {P : ℝ³} {ε θ θ_ φ φ_ : ℝ} (hP : ||P|| ≤ 1) (hε :
  ⇐ 0 < ε)
  (hθ : |θ - θ_| ≤ ε) (hφ : |φ - φ_| ≤ ε)
  (hX : √2 * ε < ⟨vecX θ_ φ_, P⟩) :
  0 < ⟨vecX θ φ, P⟩ := by
  simp using Bounding.XPgt0 hP hε hθ hφ hX
```

Lemma 3.6

Let $P \in \mathbb{R}^3$ with $\|P\| \leq 1$. Further, let $\epsilon, r > 0$ and $\bar{\theta}, \bar{\varphi}, \theta, \phi \in \mathbb{R}$ such that $|\bar{\theta} - \theta|, |\bar{\varphi} - \phi| \leq \epsilon$. If $\|M(\bar{\theta}, \bar{\varphi})P\| > r + \sqrt{2}\epsilon$ then $\|M(\theta, \phi)P\| > r$.

Proof for Lemma 3.6

See polyhedron.without.rupert, Lemma 15. Corrigendum: the triangle inequality only implies greater than **or equal to**.

Lean code for «code:lem:MPgtr»

```

theorem bp_norm_M_apply_gt {ε r θ θ_ φ φ_ : ℝ} {P : ℝ³} (hP :
  ↪ ||P|| ≤ 1) (hε : 0 < ε)
  (hθ : |θ - θ_| ≤ ε) (hφ : |φ - φ_| ≤ ε)
  (hM : r + √2 * ε < ||rotM θ_ φ_ P||) :
  r < ||rotM θ φ P|| := by
simp using Bounding.norm_M_apply_gt hP hε hθ hφ hM

```

Lemma 3.7

Let $\epsilon > 0$ and $|\theta - \bar{\theta}|, |\varphi - \bar{\varphi}|, |\alpha - \bar{\alpha}| \leq \epsilon$ then $\|R(\alpha)M(\theta, \phi) - R(\bar{\alpha})M(\bar{\theta}, \bar{\varphi})\| < \sqrt{5}\epsilon$.

Proof for Lemma 3.7

Write R_z and R_y for rotations about the third and second axes. After dropping the norm-one projection and canceling common rotations, it suffices to bound

$$\|R_z(\alpha - \bar{\alpha})R_y(\phi) - R_y(\bar{\varphi})R_z(\theta - \bar{\theta})\|.$$

Insert the ordinary midpoint $\Phi = (\phi + \bar{\varphi})/2$. On each side, the two angle changes are bounded by ϵ and $\epsilon/2$. Lemma 3.3, applied after canceling common rotations, bounds each difference strictly by $\sqrt{\epsilon^2 + (\epsilon/2)^2} = \sqrt{5}\epsilon/2$. If both angle changes vanish, strictness follows from $\epsilon > 0$; otherwise it follows from the strictness condition in that lemma. The triangle inequality gives the result. The weighted intermediate angle used in `polyhedron.without.rupert`, Lemma 16, is not needed for this uniform bound.

Lean code for «code:lem:sqrt5»

```

theorem bp_norm_RM_sub_RM_le {ε θ θ_ φ φ_ α α_ : ℝ} (hε : 0 <
  ↪ ε) (hθ : |θ - θ_| ≤ ε) (hφ : |φ - φ_| ≤ ε)
  (hα : |α - α_| ≤ ε) :
  ||rotprojRM θ φ α - rotprojRM θ_ φ_ α_|| < √5 * ε := by
simp using Bounding.norm_RM_sub_RM_le hε hθ hφ hα

```

4 Preliminaries

TODO: This whole chapter needs organization, it's just a grab bag of miscellaneous results for now.

4.1 Rupert Sets

Theorem 4.1.1

The following are equivalent:

- The convex polyhedron with vertex set v is Rupert.
- The convex closure of v is a Rupert set.

Proof for Theorem 4.1.1

TODO: import this from the other repo

4.2 Poses

TODO

Theorem 4.2.1

Given a pose with zero offset, there exists a 5-parameter pose that is equivalent to it.

Proof for Theorem 4.2.1

By putting the pose into a canonical form as a Z rotation followed by a Y followed by a Z.

4.3 Pointsymmetry and Rupertness

Theorem 4.3.1

If a set is point symmetric and convex, then it being Rupert implies it being purely rotationally Rupert.

Proof for Theorem 4.3.1

TODO: informalize proof

5 The Global Theorem

Lemma 5.1

Suppose $V = V_1, \dots, V_m \subseteq \mathbb{R}^n$ is a finite sequence of points, and let $\text{Co}(V)$ be its convex hull. If $S \in \text{Co}(V)$ and $w \in \mathbb{R}^n$, then

$$\langle S, w \rangle \leq \max_i \langle V_i, w \rangle.$$

Proof for Lemma 5.1

This is a mild generalization of Steininger and Yurkevich (2025), Lemma 18. Since $S \in \text{Co}(V)$, we have $S = \sum_{j=1}^m \lambda_j V_j$ for some $\lambda_1, \dots, \lambda_m \in [0, 1]$ with $1 = \sum_{j=1}^m \lambda_j$. Therefore

$$\begin{aligned} \langle S, w \rangle &= \left\langle \sum_{j=1}^m \lambda_j V_j, w \right\rangle = \sum_{j=1}^m \lambda_j \langle V_j, w \rangle \leq \sum_{j=1}^m \lambda_j \max_i \langle V_i, w \rangle \\ &= \max_i \langle V_i, w \rangle \sum_{j=1}^m \lambda_j = \max_i \langle V_i, w \rangle \end{aligned}$$

as required.

Lemma 5.2

Let $S \in \mathbb{R}^3$, let $w \in \mathbb{R}^2$ be a unit vector, and set $f(x_1, x_2, x_3) = \langle R(x_3)M(x_1, x_2)S, w \rangle$. Then for all $x_1, x_2, x_3 \in \mathbb{R}$ and any $i, j, k \in \{1, 2, 3\}$ it holds that

$$\left| \frac{\partial^2 f}{\partial x_i \partial x_j}(x_1, x_2, x_3) \right| \leq \|S\| \quad \text{and} \quad \left| \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(x_1, x_2, x_3) \right| \leq \|S\|.$$

The source reference normalizes f by $1/\|S\|$ and bounds the partials by 1; here f remains unnormalized and the bound carries $\|S\|$, which composes with Lemma 5.3 at $M = \|S\| \leq 1$.

Proof for Lemma 5.2

The second-partial bound is `polyhedron.without.rupert`, Lemma 19. For the third partials (and, in the formalization, every higher order at once) we argue by closure instead of enumeration: the family of functions $\pm \langle (\pm R/R')(\alpha)(\partial_\theta^a \partial_\phi^b M)(\theta, \phi)S, w \rangle$ with $a, b \leq 2$ is closed under all three partial derivatives. An α -derivative steps the head around its four-cycle $R \rightarrow R' \rightarrow -R \rightarrow -R'$, and a θ - or ϕ -derivative bumps one grid index, folding back with a sign at the edge since $\partial_\theta^3 = -\partial_\theta$ and $\partial_\phi^3 = -\partial_\phi$. Every member is bounded by $\|S\|$ because each factor has operator norm at most one, so iterating the closure bounds all iterated partials.

Lemma 5.3

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^3 -function, let $\varepsilon_1, \dots, \varepsilon_n \geq 0$, and let $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}$ satisfy $|x_i - y_i| \leq \varepsilon_i$ for all i . If, for some $M \in \mathbb{R}$, $|\partial_{x_i} \partial_{x_j} \partial_{x_k} f(v)| \leq M$ for all $i, j, k \in \{1, \dots, n\}$ and all $v \in \mathbb{R}^n$, then

$$|f(x) - f(y)| \leq \sum_{i=1}^n \varepsilon_i |\partial_{x_i} f(x)| + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \varepsilon_i \varepsilon_j |\partial_{x_i} \partial_{x_j} f(x)| + \frac{M}{6} \left(\sum_{i=1}^n \varepsilon_i \right)^3.$$

At $M = 1$ and $\varepsilon_1 = \dots = \varepsilon_n = \varepsilon$, this recovers the isotropic remainder $\frac{n^3}{6} \varepsilon^3$.

Proof for Lemma 5.3

This strengthens `polyhedron.without.rupert`, Lemma 20, by one Taylor order and per-axis radii: expand $g(t) = f((1-t)x + ty)$ to second order with Lagrange remainder, bound $|g'(0)|$ by the first sum, $|g''(0)|/2$ by the second (the second partials are evaluated exactly at x , not bounded), and $|g'''(c)| \leq M(\sum_i \varepsilon_i)^3$ using the third-partial bound.

Lemma 5.4

The partial derivatives of all relevant rotations, projections, and inner products used in the Global Theorem are as expected. Specifically:

- $f^\alpha(\theta, \phi, \alpha) = \langle R'(\alpha)M(\theta, \phi)S, w \rangle$
- $f^\theta(\theta, \phi, \alpha) = \langle R(\alpha)M^\theta(\theta, \phi)S, w \rangle$
- $f^\phi(\theta, \phi, \alpha) = \langle R(\alpha)M^\phi(\theta, \phi)S, w \rangle$
- $g^\theta(\theta, \phi) = \langle M^\theta(\theta, \phi)P, w \rangle$
- $g^\phi(\theta, \phi) = \langle M^\phi(\theta, \phi)P, w \rangle$

where $f(\theta, \phi, \alpha) = \langle R(\alpha)M(\theta, \phi)S, w \rangle$ and $g(\theta, \phi) = \langle M(\theta, \phi)P, w \rangle$.

Proof for Lemma 5.4

By basic properties of derivatives.

Theorem 5.5

Let $\mathbf{P}$ be a pointsymmetric convex polyhedron with radius $\rho = 1$ and let $S \in \mathbf{P}$. Further let $\bar{\theta}_1, \bar{\varphi}_1, \bar{\theta}_2, \bar{\varphi}_2, \bar{\alpha} \in \mathbb{R}$, let $\epsilon_\alpha, \epsilon_{\theta_1}, \epsilon_{\phi_1}, \epsilon_{\theta_2}, \epsilon_{\phi_2} \geq 0$ be per-axis radii, and let $w \in \mathbb{R}^2$ be a unit vector. Denote $\bar{M}_1 := M(\bar{\theta}_1, \bar{\varphi}_1)$, $\bar{M}_2 := M(\bar{\theta}_2, \bar{\varphi}_2)$ as well as $\bar{M}_1^\theta := M^\theta(\bar{\theta}_1, \bar{\varphi}_1)$, $\bar{M}_1^\phi := M^\phi(\bar{\theta}_1, \bar{\varphi}_1)$ and analogously for $\bar{M}_2^\theta, \bar{M}_2^\phi$. Finally set

$$\begin{aligned} G &:= \langle R(\bar{\alpha})\bar{M}_1 S, w \rangle - \epsilon_\alpha |\langle R'(\bar{\alpha})\bar{M}_1 S, w \rangle| - \epsilon_{\theta_1} |\langle R(\bar{\alpha})\bar{M}_1^\theta S, w \rangle| - \epsilon_{\phi_1} |\langle R(\bar{\alpha})\bar{M}_1^\phi S, w \rangle| \\ &\quad - \frac{1}{2} (\epsilon_\alpha^2 |\langle R(\bar{\alpha})\bar{M}_1 S, w \rangle| + 2\epsilon_\alpha \epsilon_{\theta_1} |\langle R'(\bar{\alpha})\bar{M}_1^\theta S, w \rangle| + 2\epsilon_\alpha \epsilon_{\phi_1} |\langle R'(\bar{\alpha})\bar{M}_1^\phi S, w \rangle| \\ &\quad \quad + \epsilon_{\theta_1}^2 |\langle R(\bar{\alpha})\bar{M}_1^{\theta\theta} S, w \rangle| + 2\epsilon_{\theta_1} \epsilon_{\phi_1} |\langle R(\bar{\alpha})\bar{M}_1^{\theta\phi} S, w \rangle| + \epsilon_{\phi_1}^2 |\langle R(\bar{\alpha})\bar{M}_1^{\phi\phi} S, w \rangle|) \\ &\quad - \frac{(\epsilon_\alpha + \epsilon_{\theta_1} + \epsilon_{\phi_1})^3}{6}, \\ H_P &:= \langle \bar{M}_2 P, w \rangle + \epsilon_{\theta_2} |\langle \bar{M}_2^\theta P, w \rangle| + \epsilon_{\phi_2} |\langle \bar{M}_2^\varphi P, w \rangle| \\ &\quad + \frac{1}{2} (\epsilon_{\theta_2}^2 |\langle \bar{M}_2^{\theta\theta} P, w \rangle| + 2\epsilon_{\theta_2} \epsilon_{\phi_2} |\langle \bar{M}_2^{\theta\phi} P, w \rangle| + \epsilon_{\phi_2}^2 |\langle \bar{M}_2^{\phi\phi} P, w \rangle|) + \frac{(\epsilon_{\theta_2} + \epsilon_{\phi_2})^3}{6}, \quad \text{for } P \in \mathbf{P}. \end{aligned}$$

If $G > \max_{P \in \mathbf{P}} H_P$ then there does not exist a solution to Rupert's condition with

$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in U := [\bar{\theta}_1 \pm \epsilon_{\theta_1}] \times [\bar{\varphi}_1 \pm \epsilon_{\phi_1}] \times [\bar{\theta}_2 \pm \epsilon_{\theta_2}] \times [\bar{\varphi}_2 \pm \epsilon_{\phi_2}] \times [\bar{\alpha} \pm \epsilon_\alpha] \subseteq \mathbb{R}^5.$$

This is a second-order, anisotropic strengthening of polyhedron.without.rupert, Theorem 17: the exact second partials at the center pose are charged with per-axis weights, and only the cubic Lagrange remainders are bounded via Lemma 5.2.

Proof for Theorem 5.5

See polyhedron.without.rupert, Section 4.2.

6 The Local Theorem

Lemma 6.1

For any $P \in \mathbb{R}^3$ one has $\|M(\theta, \phi)P\|^2 = \|P\|^2 - \langle X(\theta, \varphi), P \rangle^2$.

Proof for Lemma 6.1

See polyhedron.without.rupert, Lemma 21.

Definition 6.2

Given $v_1, \dots, v_n \in \mathbb{R}^n$ write $\text{span}^+(v_1, \dots, v_n)$ for the set (simplicial cone) in $\mathbb{R}^n$ defined by

$$\text{span}^+(v_1, \dots, v_n) = \left\{ w \in \mathbb{R}^n : \exists \lambda_1, \dots, \lambda_n > 0 \text{ s.t. } w = \sum_{i=1}^n \lambda_i v_i \right\},$$

which is the natural restriction of $\text{span}(v_1, \dots, v_n)$ to positive weights.

Lemma 6.3

Let $V_1, V_2, V_3, Y, Z \in \mathbb{R}^3$ with $\|Y\| = \|Z\|$ and $Z \in \text{span}^+(V_1, V_2, V_3)$. Then there exists an i with $\langle V_i, Y \rangle \leq \langle V_i, Z \rangle$.

Proof for Lemma 6.3

Write $Z = \sum_i c_i V_i$ with all $c_i > 0$. If every claimed inequality failed, then $\langle Z, Z \rangle < \langle Z, Y \rangle$ by taking the positive weighted sum. But Cauchy-Schwarz gives $\langle Z, Y \rangle \leq \|Z\| \|Y\| = \|Z\|^2$, a contradiction. Thus the assumption that Y also lies in the positive cone in Lemma 23 of polyhedron.without.rupert is unnecessary.

Lemma 6.4

For $A, \bar{A}, B, \bar{B} \in \mathbb{R}^{m \times n}$ and $P_1, P_2 \in \mathbb{R}^n$ it holds that

$$|\langle AP_1, BP_2 \rangle - \langle \bar{A}P_1, \bar{B}P_2 \rangle| \leq \|P_1\| \|P_2\| (\|A - \bar{A}\| \|\bar{B}\| + \|\bar{A}\| \|B - \bar{B}\| + \|A - \bar{A}\| \|B - \bar{B}\|).$$

Proof for Lemma 6.4

See polyhedron.without.rupert, Lemma 24.

Lemma 6.5

For $A, B \in \mathbb{R}^{m \times n}$ and $P_1, P_2 \in \mathbb{R}^n$ one has

$$|\langle AP_1, AP_2 \rangle - \langle BP_1, BP_2 \rangle| \leq \|P_1\| \|P_2\| \|A - B\| (\|A\| + \|B\| + \|A - B\|).$$

Proof for Lemma 6.5

See polyhedron.without.rupert, Lemma 25.

Lemma 6.6

Let $A, B, C \in \mathbb{R}^2$ be such that $\langle R(\pi/2)A, B \rangle, \langle R(\pi/2)B, C \rangle, \langle R(\pi/2)C, A \rangle > 0$. Then the origin lies strictly in triangle ABC .

Proof for Lemma 6.6

See polyhedron.without.rupert, Lemma 26.

Definition 6.7

Let $\theta, \varphi \in \mathbb{R}, \varepsilon > 0$, and set $M := M(\theta, \varphi)$. Three points $P_1, P_2, P_3 \in \mathbb{R}^3$ with $\|P_1\|, \|P_2\|, \|P_3\| \leq 1$ are called ε -spanning for (θ, φ) if:

- $\langle R(\pi/2)MP_1, MP_2 \rangle > 2\varepsilon(\sqrt{2} + \varepsilon)$
- $\langle R(\pi/2)MP_2, MP_3 \rangle > 2\varepsilon(\sqrt{2} + \varepsilon)$
- $\langle R(\pi/2)MP_3, MP_1 \rangle > 2\varepsilon(\sqrt{2} + \varepsilon)$

Lemma 6.8

Let $P_1, P_2, P_3 \in \mathbb{R}^3$ with $\|P_1\|, \|P_2\|, \|P_3\| \leq 1$ be ε -spanning for $(\bar{\theta}, \bar{\phi})$ and let $\theta, \phi \in \mathbb{R}$ satisfy $|\theta - \bar{\theta}|, |\phi - \bar{\phi}| \leq \varepsilon$. If $\langle X(\theta, \phi), P_i \rangle > 0$ for $i = 1, 2, 3$, then $X(\theta, \phi) \in \text{span}^+(P_1, P_2, P_3)$.

Proof for Lemma 6.8

See polyhedron.without.rupert, Lemma 28.

Lemma 6.9

Let $P, Q \in \mathbb{R}^3$ with $\|P\|, \|Q\| \leq 1$. Let $\varepsilon > 0$ and $\bar{\theta}_1, \bar{\phi}_1, \bar{\theta}_2, \bar{\phi}_2, \bar{\alpha} \in \mathbb{R}$ satisfy

$$\|R(\bar{\alpha})M(\bar{\theta}_1, \bar{\phi}_1)P - M(\bar{\theta}_2, \bar{\phi}_2)Q\| \leq 2\delta.$$

If $|\bar{\theta}_1 - \theta_1|, |\bar{\phi}_1 - \phi_1|, |\bar{\theta}_2 - \theta_2|, |\bar{\phi}_2 - \phi_2|, |\bar{\alpha} - \alpha| \leq \varepsilon$, then $\|R(\alpha)M(\theta_1, \phi_1)P - M(\theta_2, \phi_2)Q\| < 2(\delta + \sqrt{5}\varepsilon)$.

Proof for Lemma 6.9

See polyhedron.without.rupert, Lemma 30, which states that both points lie in the disc of radius $\delta + \sqrt{5}\epsilon$ around their midpoint at the reference pose; here the two memberships are combined into a single bound on the distance between the two points.

Definition 6.10

Let $\mathcal{P} \subset \mathbb{R}^2$ be a convex polygon and $Q \in \mathcal{P}$ one of its vertices. Define $\text{Sect}_\delta(Q) := \text{Disc}_\delta(Q) \cap \mathcal{P}^\circ$ as the intersection between $\text{Disc}_\delta(Q)$ and the interior of the convex hull of $\mathcal{P}$. Moreover, $Q \in \mathcal{P}$ is called δ -locally maximally distant (δ -LMD) if for all $A \in \text{Sect}_\delta(Q)$ it holds that $\|Q\| > \|A\|$. This centers the disc at Q itself; its radius plays the role of twice the radius in Definition 31 of polyhedron.without.rupert.

Lemma 6.11

Let $\mathbf{P}$ be a convex polygon and $Q \in \mathbf{P}$ a vertex, and let $\delta > 0$. Assume there exists $r > 0$ with $\|Q\| > r$ such that

$$\frac{\langle Q, Q - P_j \rangle}{\|Q\| \|Q - P_j\|} \geq \delta/r$$

for all other vertices $P_j \in \mathbf{P} \setminus Q$. Then Q is 2δ -locally maximally distant.

Proof for Lemma 6.11

See polyhedron.without.rupert, Lemma 32: the disc of radius 2δ around Q contains the paper's disc of radius δ around any $\bar{Q}$ with $\|Q - \bar{Q}\| < \delta$ (cf. Definition).

Lemma 6.12

Let $\epsilon > 0$ and $\theta, \bar{\theta}, \phi, \bar{\phi} \in \mathbb{R}$ with $|\theta - \bar{\theta}|, |\phi - \bar{\phi}| \leq \epsilon$. Define $M = M(\theta, \phi)$ and $\bar{M} = M(\bar{\theta}, \bar{\phi})$, and let $P, Q \in \mathbb{R}^3$ with $\|P\|, \|Q\| \leq 1$. Assume that

$$\frac{\langle \bar{M}P, \bar{M}(P - Q) \rangle - 2\epsilon\|P - Q\| \cdot (\sqrt{2} + \epsilon)}{(\|\bar{M}P\| + \sqrt{2}\epsilon) \cdot (\|\bar{M}(P - Q)\| + 2\sqrt{2}\epsilon)} > 0.$$

Then:

$$\frac{\langle MP, M(P - Q) \rangle}{\|MP\| \|M(P - Q)\|} \geq \frac{\langle \bar{M}P, \bar{M}(P - Q) \rangle - 2\epsilon\|P - Q\| \cdot (\sqrt{2} + \epsilon)}{(\|\bar{M}P\| + \sqrt{2}\epsilon) \cdot (\|\bar{M}(P - Q)\| + 2\sqrt{2}\epsilon)}.$$

Proof for Lemma 6.12

See polyhedron.without.rupert, Lemma 33.

Lemma 6.13

Let $P_1, P_2, P_3, Q_1, Q_2, Q_3 \in \mathbb{R}^3$. Define $P := (P_1|P_2|P_3)$ and $Q := (Q_1|Q_2|Q_3)$ and assume Q is invertible. Then P_1, P_2, P_3 and Q_1, Q_2, Q_3 are congruent iff $P^t P = Q^t Q$.

Proof for Lemma 6.13

An isometry preserves inner products, so congruence implies equality of the Gram matrices. Conversely, since the Q_i form a basis, define a linear map L by $L(Q_i) = P_i$. The equality $P^t P = Q^t Q$ says that $\langle L(Q_i), L(Q_j) \rangle = \langle Q_i, Q_j \rangle$ for every i, j . Expanding arbitrary vectors in this basis shows that L preserves all inner products, and hence is the required linear isometry. For the same construction in matrix form, if a congruence L is given then $\langle P_i, P_j \rangle = \langle LQ_i, LQ_j \rangle = \langle Q_i, Q_j \rangle$, so $P^t P = Q^t Q$. Conversely, with $L := PQ^{-1}$, we have $L^t L = (PQ^{-1})^t (PQ^{-1}) = (Q^t)^{-1} P^t P Q^{-1} = \text{Id}$ and $LQ = PQ^{-1}Q = P$, so $LQ_i = P_i$ for each i .

Theorem 6.14

Let $\mathbf{P}$ be a polyhedron with radius $\rho = 1$ and $P_1, P_2, P_3, Q_1, Q_2, Q_3 \in \mathbf{P}$ be not necessarily distinct. Assume that P_1, P_2, P_3 and Q_1, Q_2, Q_3 are congruent. Let $\epsilon > 0$ and $\bar{\theta}_1, \bar{\varphi}_1, \bar{\theta}_2, \bar{\varphi}_2, \bar{\alpha} \in \mathbb{R}$, then set $\bar{X}_1 := X(\bar{\theta}_1, \bar{\varphi}_1)$, $\bar{X}_2 := X(\bar{\theta}_2, \bar{\varphi}_2)$ as well as $\bar{M}_1 := M(\bar{\theta}_1, \bar{\varphi}_1)$, $\bar{M}_2 := M(\bar{\theta}_2, \bar{\varphi}_2)$. Assume that there exist $\sigma_P, \sigma_Q \in \{0, 1\}$ such that

$$(-1)^{\sigma_P} \langle \bar{X}_1, P_i \rangle > \sqrt{2}\epsilon \quad \text{and} \quad (-1)^{\sigma_Q} \langle \bar{X}_2, Q_i \rangle > \sqrt{2}\epsilon,$$

for all $i = 1, 2, 3$. Moreover, assume that P_1, P_2, P_3 are ϵ -spanning for $(\bar{\theta}_1, \bar{\varphi}_1)$ and that Q_1, Q_2, Q_3 are ϵ -spanning for $(\bar{\theta}_2, \bar{\varphi}_2)$. Finally, assume that for all $i = 1, 2, 3$ and any $Q_j \in \mathbf{P} \setminus Q_i$ it holds that

$$\frac{\langle \bar{M}_2 Q_i, \bar{M}_2(Q_i - Q_j) \rangle - 2\epsilon \|Q_i - Q_j\| \cdot (\sqrt{2} + \epsilon)}{(\|\bar{M}_2 Q_i\| + \sqrt{2}\epsilon) \cdot (\|\bar{M}_2(Q_i - Q_j)\| + 2\sqrt{2}\epsilon)} > \frac{\sqrt{5}\epsilon + \delta}{r},$$

for some $r > 0$ such that $\min_{i=1,2,3} \|\bar{M}_2 Q_i\| > r + \sqrt{2}\epsilon$ and for some $\delta \in \mathbb{R}$ with

$$\delta \geq \max_{i=1,2,3} \|R(\bar{\alpha})\bar{M}_1 P_i - \bar{M}_2 Q_i\| / 2.$$

Then there exists no solution to Rupert's problem $R(\alpha)M(\theta_1, \phi_1)\mathbf{P} \subset M(\theta_2, \phi_2)\mathbf{P}^\circ$ with

$$(\theta_1, \phi_1, \theta_2, \phi_2, \alpha) \in [\bar{\theta}_1 \pm \epsilon, \bar{\varphi}_1 \pm \epsilon, \bar{\theta}_2 \pm \epsilon, \bar{\varphi}_2 \pm \epsilon, \bar{\alpha} \pm \epsilon] := U \subseteq \mathbb{R}^5.$$

Proof for Theorem 6.14

See polyhedron.without.rupert, Theorem 36.

7 Rational Versions

Definition 7.1

Define $\sin_{\mathbb{Q}}, \cos_{\mathbb{Q}} : \mathbb{R} \rightarrow \mathbb{R}$ by truncated Taylor series:

- $\sin_{\mathbb{Q}}(x) := x - \frac{x^3}{3!} + \frac{x^5}{5!} \mp \cdots + \frac{x^{25}}{25!}$
- $\cos_{\mathbb{Q}}(x) := 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \mp \cdots + \frac{x^{24}}{24!}$

Replacing $\sin, \cos$ with $\sin_{\mathbb{Q}}, \cos_{\mathbb{Q}}$ defines $R_{\mathbb{Q}}(\alpha), R'_{\mathbb{Q}}(\alpha), X_{\mathbb{Q}}(\theta, \phi), M_{\mathbb{Q}}(\theta, \phi), M_{\mathbb{Q}}^{\theta}(\theta, \varphi), M_{\mathbb{Q}}^{\phi}(\theta, \varphi)$.

Lemma 7.2

$$|\sin_{\mathbb{Q}}(x) - \sin(x)| \leq \frac{|x|^{27}}{27!} \quad \text{and} \quad |\cos_{\mathbb{Q}}(x) - \cos(x)| \leq \frac{|x|^{26}}{26!}.$$

Proof for Lemma 7.2

Appeal to Taylor series bounds, using the fact that all absolute values of higher derivatives of sine and cosine never exceed 1.

Lemma 7.3

For every $x \in [-4, 4]$, $|\sin_{\mathbb{Q}}(x) - \sin(x)| \leq \kappa/7$ and $|\cos_{\mathbb{Q}}(x) - \cos(x)| \leq \kappa/7$.

Proof for Lemma 7.3

Straightforward numerical calculation from Lemma Lemma 7.2.

Lemma 7.4

Let $A = (a_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n} \in \mathbb{R}^{m \times n}$ and $\delta > 0$. If $|a_{i,j}| \leq \delta$ for all entries, then $\|A\| \leq \delta\sqrt{mn}$.

Proof for Lemma 7.4

For any $v \in \mathbb{R}^n$ we have

$$\|Av\|^2 = \sum_{i=1}^m \left(\sum_{j=1}^n a_{i,j} v_j \right)^2 \leq \sum_{i=1}^m \left(\sum_{j=1}^n \delta |v_j| \right)^2 = \delta^2 m \left(\sum_{j=1}^n |v_j| \right)^2 \leq \delta^2 mn \|v\|^2$$

using the Cauchy-Schwarz inequality. Dividing by $\|v\|$ and taking the square root proves the claim.

Lemma 7.5

Let $A(x, y)$ be an $m \times n$ matrix ($1 \leq m, n \leq 3$) whose entries are products of terms in $\{0, 1, -1, \pm \sin(z), \pm \cos(z)\}$. Let $A_{\mathbb{Q}}(x, y)$ be obtained by replacing $\sin, \cos$ with $\sin_{\mathbb{Q}}, \cos_{\mathbb{Q}}$. Then for $x, y \in [-4, 4]$, $\|A(x, y) - A_{\mathbb{Q}}(x, y)\| \leq \kappa$.

Proof for Lemma 7.5

We've replaced the assumption $a_i(z) \in \{0, 1, -1, \pm \sin(z), \pm \cos(z)\}$ in Steining and Yurkevich (2025)'s Lemma 40 with $a_i(z) \in [-1, 1]$. By assumption, for fixed x, y every entry of $A(x, y) - A_{\mathbb{Q}}(x, y)$ is of the form $ab - \tilde{a}\tilde{b}$ for some $a, b \in [-1, 1]$ and $|a - \tilde{a}|, |b - \tilde{b}| \leq \kappa/7$ by Lemma 7.3. This implies that

$$\begin{aligned} |ab - \tilde{a}\tilde{b}| &\leq |ab - a\tilde{b}| + |a\tilde{b} - \tilde{a}\tilde{b}| \\ &= |a| \cdot |b - \tilde{b}| + |\tilde{b}| \cdot |a - \tilde{a}| \leq 1 \cdot \kappa/7 + (1 + \kappa/7) \cdot \kappa/7 < \kappa/3. \end{aligned}$$

So we can apply Lemma 7.4 and obtain that $\|A(x, y) - A_{\mathbb{Q}}(x, y)\| < \kappa/3 \cdot \sqrt{3 \cdot 3} = \kappa$.

Corollary 7.6

Let $\alpha, \theta, \phi \in [-4, 4]$. Then $\|R(\alpha) - R_{\mathbb{Q}}(\alpha)\|, \|R'(\alpha) - R'_{\mathbb{Q}}(\alpha)\|, \|X(\theta, \phi) - X_{\mathbb{Q}}(\theta, \phi)\|, \|M(\theta, \phi) - M_{\mathbb{Q}}(\theta, \phi)\|, \|M^{\theta}(\theta, \phi) - M_{\mathbb{Q}}^{\theta}(\theta, \phi)\|, \|M^{\phi}(\theta, \phi) - M_{\mathbb{Q}}^{\phi}(\theta, \phi)\|, \|M^{\theta\theta}(\theta, \phi) - M_{\mathbb{Q}}^{\theta\theta}(\theta, \phi)\|, \|M^{\theta\phi}(\theta, \phi) - M_{\mathbb{Q}}^{\theta\phi}(\theta, \phi)\|, \|M^{\phi\phi}(\theta, \phi) - M_{\mathbb{Q}}^{\phi\phi}(\theta, \phi)\|$ are all at most κ . Moreover the corresponding rational matrices $R_{\mathbb{Q}}, R'_{\mathbb{Q}}, M_{\mathbb{Q}}, M_{\mathbb{Q}}^{\theta}, M_{\mathbb{Q}}^{\phi}, M_{\mathbb{Q}}^{\theta\theta}, M_{\mathbb{Q}}^{\theta\phi}, M_{\mathbb{Q}}^{\phi\phi}$ have norm at most $1 + \kappa$.

Proof for Corollary 7.6

The first statement is a direct application of Lemma 7.5 and the second statement follows immediately after using Lemma 3.1 and the triangle inequality. The derivative norm bounds follow similarly, using that the operator norms of R', M^{θ} , and M^{ϕ} are at most 1.

Lemma 7.7

Let (A_i, B_i) for $1 \leq i \leq n$ be pairs of same-size real matrices, with products $A_1 \cdots A_n$ and $B_1 \cdots B_n$ defined. If $\|A_i - B_i\| \leq \kappa$ and $\delta_i \geq \max(\|A_i\|, \|B_i\|, 1)$, then $\|A_1 \cdots A_n - B_1 \cdots B_n\| \leq n\kappa \delta_1 \cdots \delta_n$.

Proof for Lemma 7.7

See polyhedron.without.rupert, Lemma 42.

Lemma 7.8

Let $\alpha, \theta, \phi \in [-4, 4]$, $P \in \mathbb{R}^3$ with $\|P\| \leq 1$ and let $\tilde{P}$ be a κ -rational approximation of P . Set $M = M(\theta, \phi)$ and $M_{\mathbb{Q}} = M_{\mathbb{Q}}(\theta, \phi)$, $M^{\theta} = M^{\theta}(\theta, \phi)$ and $M_{\mathbb{Q}}^{\theta} = M_{\mathbb{Q}}^{\theta}(\theta, \phi)$, $M^{\phi} = M^{\phi}(\theta, \phi)$ and $M_{\mathbb{Q}}^{\phi} = M_{\mathbb{Q}}^{\phi}(\theta, \phi)$, as well as $R = R(\alpha)$, $R_{\mathbb{Q}} = R_{\mathbb{Q}}(\alpha)$, $R' = R'(\alpha)$, $R'_{\mathbb{Q}} = R'_{\mathbb{Q}}(\alpha)$. Finally let $w \in \mathbb{R}^2$ with $\|w\| = 1$. Then:

$$|\langle MP, w \rangle - \langle M_{\mathbb{Q}} \tilde{P}, w \rangle| \leq 3\kappa \quad (7.1)$$

$$|\langle M^{\theta} P, w \rangle - \langle M_{\mathbb{Q}}^{\theta} \tilde{P}, w \rangle| \leq 3\kappa, \quad (7.2)$$

$$|\langle M^{\phi} P, w \rangle - \langle M_{\mathbb{Q}}^{\phi} \tilde{P}, w \rangle| \leq 3\kappa, \quad (7.3)$$

$$|\langle RMP, w \rangle - \langle R_{\mathbb{Q}} M_{\mathbb{Q}} \tilde{P}, w \rangle| \leq 4\kappa \quad (7.4)$$

$$|\langle R'MP, w \rangle - \langle R'_{\mathbb{Q}} M_{\mathbb{Q}} \tilde{P}, w \rangle| \leq 4\kappa, \quad (7.5)$$

$$|\langle RM^{\theta} P, w \rangle - \langle R_{\mathbb{Q}} M_{\mathbb{Q}}^{\theta} \tilde{P}, w \rangle| \leq 4\kappa, \quad (7.6)$$

$$|\langle RM^{\phi} P, w \rangle - \langle R_{\mathbb{Q}} M_{\mathbb{Q}}^{\phi} \tilde{P}, w \rangle| \leq 4\kappa. \quad (7.7)$$

The same bounds hold with M replaced by any of the second-derivative matrices $M^{\theta\theta}$, $M^{\theta\phi}$, or $M^{\phi\phi}$ (at 3κ without a rotation factor and 4κ with R in front), and with R' in place of R in front of M^{θ} and M^{ϕ} .

Proof for Lemma 7.8

See polyhedron.without.rupert, Lemma 44.

Theorem 7.9

Let $\mathbf{P}$ be a pointsymmetric convex polyhedron with radius $\rho = 1$ and $\tilde{\mathbf{P}}$ a κ -rational approximation. Let $\tilde{S} \in \tilde{\mathbf{P}}$. Further let $\epsilon_{\alpha}, \epsilon_{\theta_1}, \epsilon_{\phi_1}, \epsilon_{\theta_2}, \epsilon_{\phi_2} \in \mathbb{Q}_{\geq 0}$ be per-axis radii and let $\bar{\theta}_1, \bar{\varphi}_1, \bar{\theta}_2, \bar{\varphi}_2, \bar{\alpha} \in \mathbb{Q} \cap [-4, 4]$. Let $w \in \mathbb{Q}^2$ be a unit vector. Denote $\bar{M}_1 := M_{\mathbb{Q}}(\theta_1, \varphi_1)$, $\bar{M}_2 := M_{\mathbb{Q}}(\theta_2, \varphi_2)$ as well as $\bar{M}_1^{\theta} := M_{\mathbb{Q}}^{\theta}(\bar{\theta}_1, \bar{\varphi}_1)$, $\bar{M}_1^{\phi} := M_{\mathbb{Q}}^{\phi}(\bar{\theta}_1, \bar{\varphi}_1)$ and analogously

for $\overline{M}_2^\theta, \overline{M}_2^\phi$. Write $E_1 := \epsilon_\alpha + \epsilon_{\theta_1} + \epsilon_{\phi_1}$ and $E_2 := \epsilon_{\theta_2} + \epsilon_{\phi_2}$, and set

$$\begin{aligned} G^\mathbb{Q} &:= \langle R_\mathbb{Q}(\overline{\alpha})\overline{M}_1\tilde{S}, w \rangle - \epsilon_\alpha |\langle R'_\mathbb{Q}(\overline{\alpha})\overline{M}_1\tilde{S}, w \rangle| - \epsilon_{\theta_1} |\langle R_\mathbb{Q}(\overline{\alpha})\overline{M}_1^{\theta_1}\tilde{S}, w \rangle| - \epsilon_{\phi_1} |\langle R_\mathbb{Q}(\overline{\alpha})\overline{M}_1^{\phi_1}\tilde{S}, w \rangle| \\ &\quad - \frac{1}{2} (\epsilon_\alpha^2 |\langle R_\mathbb{Q}(\overline{\alpha})\overline{M}_1\tilde{S}, w \rangle| + 2\epsilon_\alpha\epsilon_{\theta_1} |\langle R'_\mathbb{Q}(\overline{\alpha})\overline{M}_1^{\theta_1}\tilde{S}, w \rangle| + 2\epsilon_\alpha\epsilon_{\phi_1} |\langle R'_\mathbb{Q}(\overline{\alpha})\overline{M}_1^{\phi_1}\tilde{S}, w \rangle| \\ &\quad + \epsilon_{\theta_1}^2 |\langle R_\mathbb{Q}(\overline{\alpha})\overline{M}_1^{\theta\theta}\tilde{S}, w \rangle| + 2\epsilon_{\theta_1}\epsilon_{\phi_1} |\langle R_\mathbb{Q}(\overline{\alpha})\overline{M}_1^{\theta\phi}\tilde{S}, w \rangle| + \epsilon_{\phi_1}^2 |\langle R_\mathbb{Q}(\overline{\alpha})\overline{M}_1^{\phi\phi}\tilde{S}, w \rangle|) \\ &\quad - \frac{E_1^3}{6} - 4\kappa(1 + E_1 + \frac{1}{2}E_1^2), \\ H_P^\mathbb{Q} &:= \langle \overline{M}_2 P, w \rangle + \epsilon_{\theta_2} |\langle \overline{M}_2^{\theta_2} P, w \rangle| + \epsilon_{\phi_2} |\langle \overline{M}_2^{\phi_2} P, w \rangle| \\ &\quad + \frac{1}{2} (\epsilon_{\theta_2}^2 |\langle \overline{M}_2^{\theta\theta} P, w \rangle| + 2\epsilon_{\theta_2}\epsilon_{\phi_2} |\langle \overline{M}_2^{\theta\phi} P, w \rangle| + \epsilon_{\phi_2}^2 |\langle \overline{M}_2^{\phi\phi} P, w \rangle|) + \frac{E_2^3}{6} + 3\kappa(1 + E_2 + \frac{1}{2}E_2^2). \end{aligned}$$

If $G^\mathbb{Q} > \max_{P \in \tilde{\mathbf{P}}} H_P^\mathbb{Q}$ then there does not exist a solution to Rupert's condition to $\mathbf{P}$ with

$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\bar{\theta}_1 \pm \epsilon_{\theta_1}] \times [\bar{\varphi}_1 \pm \epsilon_{\phi_1}] \times [\bar{\theta}_2 \pm \epsilon_{\theta_2}] \times [\bar{\varphi}_2 \pm \epsilon_{\phi_2}] \times [\bar{\alpha} \pm \epsilon_\alpha].$$

On the diagonal $\epsilon_\alpha = \dots = \epsilon$, the rational slack terms become $4\kappa(1 + 3\epsilon + \frac{9}{2}\epsilon^2)$ and $3\kappa(1 + 2\epsilon + 2\epsilon^2)$.

Proof for Theorem 7.9

Definition 7.10

Let $\theta, \phi \in \mathbb{Q} \cap [-4, 4]$ and $M_\mathbb{Q} := M_\mathbb{Q}(\theta, \phi)$. Three points $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3 \in \mathbb{Q}^3$ with $\|\tilde{P}_1\|, \|\tilde{P}_2\|, \|\tilde{P}_3\| \leq 1 + \kappa$ are called ϵ - κ -spanning for (θ, ϕ) if it holds that:

$$\begin{aligned} \langle R(\pi/2)M_\mathbb{Q}\tilde{P}_1, M_\mathbb{Q}\tilde{P}_2 \rangle &> 2\epsilon(\sqrt{2} + \epsilon) + 6\kappa, \\ \langle R(\pi/2)M_\mathbb{Q}\tilde{P}_2, M_\mathbb{Q}\tilde{P}_3 \rangle &> 2\epsilon(\sqrt{2} + \epsilon) + 6\kappa, \\ \langle R(\pi/2)M_\mathbb{Q}\tilde{P}_3, M_\mathbb{Q}\tilde{P}_1 \rangle &> 2\epsilon(\sqrt{2} + \epsilon) + 6\kappa. \end{aligned}$$

Lemma 7.11

Let $P_1, P_2, P_3 \in \mathbb{R}^3$ with $\|P_i\| \leq 1$ and $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3 \in \mathbb{Q}^3$ be their κ -rational approximations. Assume that $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3$ are ϵ - κ -spanning for some $\theta, \phi \in \mathbb{Q} \cap [-4, 4]$, then P_1, P_2, P_3 are ϵ -spanning for θ, ϕ .

Proof for Lemma 7.11

See polyhedron.without.rupert, Lemma 46.

Lemma 7.12

Let $P \in \mathbb{R}^3$ with $\|P\| \leq 1$ and let $\tilde{P}$ be a κ -rational approximation of P . For $\theta, \phi \in [-4, 4]$, set $X = X(\theta, \phi)$ and $X_{\mathbb{Q}} = X_{\mathbb{Q}}(\theta, \phi)$. Then $|\langle X, P \rangle - \langle X_{\mathbb{Q}}, \tilde{P} \rangle| \leq 3\kappa$. The inner-product and norm bounds for M and $M_{\mathbb{Q}}$ from Lemma 49 of polyhedron.without.rupert are formalized only in the generalized forms used by Corollary 7.13.

Proof for Lemma 7.12

See polyhedron.without.rupert, Lemma 49.

Corollary 7.13

In the setting of Lemma 7.12, let $\sqrt[4]{x}$ be an upper square-root function, i.e., $\sqrt{x} \leq \sqrt[4]{x}$ for all real $x \geq 0$ with rational output on rational input. Set $\|x\|_+ := \sqrt[4]{\|x\|^2}$. Set

$$A = \frac{\langle MP, M(P - Q) \rangle - 2\epsilon\|P - Q\| \cdot (\sqrt{2} + \epsilon)}{(\|MP\| + \sqrt{2}\epsilon) \cdot (\|M(P - Q)\| + 2\sqrt{2}\epsilon)}$$

as well as

$$A_{\mathbb{Q}} = \frac{\langle M_{\mathbb{Q}}\tilde{P}, M_{\mathbb{Q}}(\tilde{P} - \tilde{Q}) \rangle - 10\kappa - 2\epsilon(\|\tilde{P} - \tilde{Q}\| + 2\kappa) \cdot (\sqrt{2} + \epsilon)}{(\|M_{\mathbb{Q}}\tilde{P}\|_+ + \sqrt{2}\epsilon + 3\kappa) \cdot (\|M_{\mathbb{Q}}(\tilde{P} - \tilde{Q})\|_+ + 2\sqrt{2}\epsilon + 6\kappa)}.$$

Assume that $A \geq 0$. Then $A \geq A_{\mathbb{Q}}$. In the Lean formalization, each applied vector $M_{\mathbb{Q}}\tilde{x}$ in $A_{\mathbb{Q}}$ is additionally rounded down componentwise to a multiple of 10^{-13} so that the checker's dot products and norms use small denominators; the resulting perturbation is absorbed into the existing slack terms.

Proof for Corollary 7.13

See polyhedron.without.rupert, Corollary 51.

Theorem 7.14

Let $\mathbf{P}$ be a polyhedron with radius $\rho = 1$ and $\tilde{P}_i$ be a κ -rational approximation of $P_i \in \mathbf{P}$. Set $\tilde{\mathbf{P}} = \{\tilde{P}_i \text{ for } P_i \in \mathbf{P}\}$. Let $P_1, P_2, P_3, Q_1, Q_2, Q_3 \in \mathbf{P}$ be not necessarily distinct and assume that P_1, P_2, P_3 and Q_1, Q_2, Q_3 are congruent. Let $\epsilon > 0$ and $\bar{\theta}_1, \bar{\varphi}_1, \bar{\theta}_2, \bar{\varphi}_2, \bar{\alpha} \in \mathbb{Q} \cap [-4, 4]$. Set $\bar{X}_1 := X_{\mathbb{Q}}(\bar{\theta}_1, \bar{\varphi}_1)$, $\bar{X}_2 := X_{\mathbb{Q}}(\bar{\theta}_2, \bar{\varphi}_2)$ as well as $\bar{M}_1 := M_{\mathbb{Q}}(\bar{\theta}_1, \bar{\varphi}_1)$, $\bar{M}_2 := M_{\mathbb{Q}}(\bar{\theta}_2, \bar{\varphi}_2)$. Assume that there exist $\sigma_P, \sigma_Q \in \{0, 1\}$ such that

$$(-1)^{\sigma_P} \langle \bar{X}_1, \tilde{P}_i \rangle > \sqrt{2}\epsilon + 3\kappa \quad \text{and} \quad (-1)^{\sigma_Q} \langle \bar{X}_2, \tilde{Q}_i \rangle > \sqrt{2}\epsilon + 3\kappa,$$

for all $i = 1, 2, 3$. Moreover, assume that $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3$ are ϵ - κ -spanning for $(\bar{\theta}_1, \bar{\varphi}_1)$ and that $\tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3$ are ϵ - κ -spanning for $(\bar{\theta}_2, \bar{\varphi}_2)$. Let $\sqrt[+]{x}$ and $\sqrt[-]{x}$ be upper- and lower-square-root functions, then set $\|Z\|_+ := \sqrt[+]{\|Z\|^2}$ and $\|Z\|_- := \sqrt[-]{\|Z\|^2}$ for $Z \in \mathbb{R}^n$. Finally, assume that for all $i = 1, 2, 3$ and any $\tilde{Q}_j \in \tilde{\mathbf{P}} \setminus \tilde{Q}_i$ it holds that the rational inequality $(B_\epsilon^{\mathbb{Q}})$ from the paper is satisfied, for some $r > 0$ such that $\min_{i=1,2,3} \|\overline{M_2} \tilde{Q}_i\|_- > r + \sqrt{2}\epsilon + 3\kappa$ and for some $\delta \in \mathbb{R}$ with

$$\delta \geq \max_{i=1,2,3} \left\| R_{\mathbb{Q}}(\bar{\alpha}) \overline{M_1} \tilde{P}_i - \overline{M_2} \tilde{Q}_i \right\|_+ / 2 + 3\kappa.$$

Then there exists no solution to Rupert's problem $R(\alpha)M(\theta_1, \phi_1)\mathbf{P} \subset M(\theta_2, \phi_2)\mathbf{P}^\circ$ with

$$(\theta_1, \phi_1, \theta_2, \phi_2, \alpha) \in [\bar{\theta}_1 \pm \epsilon, \bar{\varphi}_1 \pm \epsilon, \bar{\theta}_2 \pm \epsilon, \bar{\varphi}_2 \pm \epsilon, \bar{\alpha} \pm \epsilon] \subseteq \mathbb{R}^5.$$

In the Lean formalization, the applied vector $\overline{M_2} \tilde{Q}_i$ in the r -condition is additionally rounded down componentwise to a multiple of 10^{-13} ; the resulting perturbation is absorbed into the 3κ term.

Proof for Theorem 7.14

8 Computational Step

Definition 8.1

We define rational approximations of the 90 `noperthedron` vertices by $\lfloor x \cdot 10^{16} \rfloor / 10^{16}$.

Theorem 8.2

The rational vertex set `nopertQ` is a κ -rational approximation of the `Noperthedron`.

Proof for Theorem 8.2

Theorem 8.3

There exists a valid solution table whose zeroth row covers

$$\begin{aligned}\theta_1, \theta_2 &\in [0, 2\pi/15] \subset [0, 0.42], \\ \varphi_1 &\in [0, \pi] \subset [0, 3.15], \\ \varphi_2 &\in [0, \pi/2] \subset [0, 1.58], \\ \alpha &\in [-\pi/2, \pi/2] \subset [-1.58, 1.58].\end{aligned}$$

Proof for Theorem 8.3

By exhibiting the table and running the validity checking algorithm. Formalized as `Noperthedron.NativeCaseAnalysis.solutionTable` in the build-on-demand `NativeCaseAnalysis` library (checked with `native_decide`) and independently in the kernel-only `KernelCaseAnalysis` library (checked with `decide + kernel`, using only the standard axioms); both are kept out of the default build targets so that CI runs stay fast.

Theorem 8.4

If a global node in the solution tree is valid, then there is no Rupert solution for its interval.

Proof for Theorem 8.4

Theorem 8.5

If a local node in the solution tree is valid, then there is no Rupert solution for its interval.

Proof for Theorem 8.5

Theorem 8.6

If we have a valid solution table, and in particular its i th row is valid, then there is no Rupert solution of the interval of its i th row.

Proof for Theorem 8.6

By strong induction on the number of rows left in the table following the i th. This is because validity constrains each row to only refer to later entries. For a split row, every point in its interval belongs to a child interval at a larger index, so the induction hypothesis excludes a Rupert solution there. At a leaf, apply Theorem or Theorem.

Corollary 8.7

If we have a valid solution table, then there is no Rupert solution of the interval of its zeroth row.

Proof for Corollary 8.7

Immediate special case of Theorem 8.6.

9 Main Theorems

Theorem 9.1

There does not in fact exist a noperthedron Rupert solution with

$$\begin{aligned}\theta_1, \theta_2 &\in [0, 2\pi/15] \subset [0, 0.42], \\ \varphi_1 &\in [0, \pi] \subset [0, 3.15], \\ \varphi_2 &\in [0, \pi/2] \subset [0, 1.58], \\ \alpha &\in [-\pi/2, \pi/2] \subset [-1.58, 1.58].\end{aligned}$$

Proof for Theorem 9.1

By this theorem, there is a valid solution table containing a valid row whose pose interval is a superset of the 5-dimensional interval above. By this theorem, there is no Rupert solution in that interval.

Theorem 9.2

There is no 5-parameter pose that makes the noperthedron have the Rupert property.

Proof for Theorem 9.2

Theorem Theorem 9.1 says there is no tight pose that makes the noperthedron Rupert. Corollary Corollary 2.2.2 says that this suffices for the general case.

Theorem 9.3

There is no purely rotational pose that makes the noperthedron have the Rupert property.

Proof for Theorem 9.3

Suppose there were a purely rotational pose. Then convert that to an equivalent 5-parameter pose with Theorem Theorem 4.2.1 and appeal to Theorem Theorem 9.2.

Theorem 9.4

There is no pose that makes the noperthedron have the Rupert property.

Proof for Theorem 9.4

By Theorem Theorem 4.3.1, we need only show that the noperthedron is pointsymmetric to see that if it is Rupert, then it must be Rupert via a purely rotational pose. But Lemma Lemma 2.1.5 shows exactly this. And yet we know via Theorem Theorem 9.3 that the noperthedron is not rotationally Rupert, so we have a contradiction, hence the noperthedron has no pose that makes it Rupert.

Theorem 9.5

The noperthedron is not a Rupert set.

Proof for Theorem 9.5

By Theorem Theorem 9.4, there is no pose that makes the noperthedron a Rupert set.

Theorem 9.6

The noperthedron is not a Rupert polyhedron.

Proof for Theorem 9.6

By Theorem 4.1.1, it suffices to show that the convex hull of noperthedron vertices is not a Rupert set, which is exactly the previous theorem.

Dependency Graph

The dependency graph is available in the HTML output.

Blueprint Summary

The Blueprint summary is available in the HTML output.

Blueprint Bibliography

polyhedron.without.rupert Jakob Steininger and Sergey Yurkevich, 2025. “A convex polyhedron without Rupert’s property”¹. `arXiv:polyhedron.without.rupert`

¹<https://arxiv.org/abs/polyhedron.without.rupert>