Carleson Blueprint

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This repository contains the completed Verso port of `Carleson Blueprint`.

1 Formalization of Carleson's theorem

This paper is the blueprint underlying the Lean formalization of the proof of Carleson's classical result asserting almost everywhere convergence of Fourier series of continuous functions. We break up the proof into two steps, a reduction of the classical result to a new theorem that appears in the sibling communication and a proof of this new theorem, which is also detailed as blueprint in this paper. An early version of this blueprint was used to initiate the Lean formalization. During the formalization, many contributors elaborated the blueprint with minor corrections, modifications and extensions. The final version is presented here as a guide through the accompanying Lean code.

1.1 Introduction

Trigonometric series represent functions as possibly infinite linear combinations of pure frequencies. They gained particular prominence through the work of J. Fourier, who used them in his analytical theory of heat, thereby establishing them as a tool for solving partial differential equations. Fourier also made the groundbreaking claim that a wide range of functions could be represented using trigonometric series. This sparked the interest of many mathematicians, including Dirichlet, who gave some rigorous conditions for convergence of Fourier series, as trigonometric series are now called. Dirichlet also opened a branch of analytic number theory partially inspired by the ideas of Fourier. Nowadays, Fourier analysis plays an important role in many areas of mathematics.

With Euler's formula to represent pure frequencies in mind, a trigonometric polynomial can be expressed as

$$S_N(x) := \sum_{n=-N}^{N} c_n e^{inx}.$$

The Fourier series is then defined as the limit f of such a sequence S_N as N tends to ∞. Fourier's vision to represent rather general functions raises two fundamental questions. The first question is to identify the appropriate choice of coefficients c_n to use to represent a given f. The second question addresses the convergence of S_N to f.

The first question has a fairly canonical and standard answer, provided by the Fourier integral formula:

$$c_n := \widehat{f}_n := \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx.$$

Here the precise interpretation of the integral depends on the chosen theory of integration. For a continuous function f, Riemann's notion of the integral suffices. If f is integrable in the Lebesgue sense, $f \in L^1(0, 2\pi)$, the Lebesgue integral is appropriate. More generally, if f is a distribution in the sense of Schwartz, supported in $[0, 2\pi]$, the integral can be understood as an evaluation against the periodic test function e^{-inx}. In each case, the more general definition reduces to the simpler one within the respective more restrictive domain. Hence, the Fourier coefficients given above serve as a universal choice. This choice is unique in several respects, in particular if one is to exactly reproduce a trigonometric polynomial f.

The second question of convergence bifurcates into the question of pointwise convergence of the series with coefficients given above for a given x on the one hand and convergence of the functions S_N to the function f in a suitable function space with corresponding topology on the other hand. There are at least as many function spaces for the question of convergence as there are different definitions of the integral elaborated earlier. There are some very canonical answers to the convergence question in function spaces, albeit not known at the time of Fourier and Dirichlet. One example is convergence in the Hilbert space sense for f in $L^2(0, 2\pi)$, as discovered in the first decade of the twentieth century as a consequence of the rapid development of Lebesgue integration theory. Another canonical example is convergence in the sense of distributions for general distributional f, as discovered a few decades after Lebesgue integration. For some other natural spaces, such as $L^1(0, 2\pi)$, there is no guarantee of convergence in the norm of that space even if f is in the space.

In contrast to these examples of function spaces with a very natural theory of convergence of Fourier series in the topology of the function space, there are no similarly elegant solutions to the characterization of pointwise convergence. In particular, the space of functions f such that the sequence $S_N(x)$ converges for every x does not have a good characterization in terms of f itself. Similarly, the space of all functions f such that the sequence of coefficients $\widehat{f}_n$ is absolutely summable has also no good characterization.

When the Fourier integral is defined in the Lebesgue sense and $f \in L^1(0, 2\pi)$, then the function f itself is meaningful not everywhere but only pointwise almost everywhere in the Lebesgue sense. The question of pointwise convergence to f for all x then becomes meaningless, and instead one asks for almost everywhere convergence. Such convergence was conjectured by N. Luzin for the space $L^2(0, 2\pi)$. Kolmogorov's example of an L^1 function whose Fourier series diverges at almost every point seemed to provide some evidence against Luzin's conjecture. Indeed, in the 1960s, L. Carleson tried to construct a counterexample to Luzin's conjecture. In his own recollection, his efforts led him to better and better understand how such a potential counterexample should look like. In the end, the counterexample had to satisfy so many properties that the properties started to contradict each other, and Carleson realized that a counterexample could not exist. Thus, he had proved Luzin's conjecture. In particular, he had proven the more elementary statement

Theorem 1.1.1

> Let f be a 2π-periodic complex-valued continuous function on $\mathbb{R}$. Then for almost all $x \in \mathbb{R}$ we have
>
> $$\lim_{N\to\infty} S_N f(x) = f(x),$$
>
> where $S_N f$ is the N-th partial Fourier sum defined above with Fourier coefficients defined above.

Here, almost every x means in the Lebesgue sense, that is, for every $\epsilon > 0$ the set of x where convergence fails can be covered by a sequence of intervals such that the sum of the lengths of these intervals is less than ϵ. While Carleson had proven the more general Luzin conjecture for functions in $L^2[0, 2\pi]$, even the more elementary statement for continuous functions was not known before Carleson's work. Moreover, until now, the elementary statement has not seen any substantially easier proof than those generalizing to L^2, partially because there is no readily usable criterion on the level of Fourier coefficients to distinguish between continuous functions and L^2 functions.

Shortly after Carleson's breakthrough, Hunt proved the analogous result for L^p functions with $p > 1$ and for functions in $L^1 \log(L)^2$. Billard adapted Carleson's arguments to prove almost everywhere convergence of Walsh-Fourier series for functions in L^2.

In Fefferman, C. Fefferman gave an alternative proof of Carleson's theorem via an a priori bound for Carleson's operator, the maximally modulated singular integral

$$Tf(x) := \sup_N \int_{-\pi}^{\pi} e^{iNy} f(x-y) \frac{1}{y} dy.$$

In Lacey-Thiele, a duality between the approaches by Carleson and Fefferman was pointed out and a more symmetric and self-dual proof was presented.

Various strengthenings of Fefferman's estimates for Carleson's operator have appeared since, such as bounds for a higher dimensional Carleson operator in the isotropic and anisotropic setting. The supremum norm in the variable N in the displayed operator above was strengthened to maximal multiplier norms with applications to return times theorems in ergodic theory. It was strengthened to variation norms V^r with $r > 2$ with applications to non-linear analysis.

Stein and Wainger proposed to study variants of the displayed operator above replacing the linear modulation phase Ny by polynomial phases in y and proved a result for polynomials without a linear term. This was generalized by Lie to general polynomial Carleson operators, first for a supremum over all quadratic polynomials and then for a supremum over all polynomials. The polynomial Carleson operator was further generalized to higher dimensions and Holder regular kernels. The development of polynomial Carleson operators has led to further generalisations, for example to almost everywhere convergence of Malmquist-Takenaka series and maximally modulated singular Radon transforms.

Surveys to Carleson's theorem can be found in the literature.

In this blueprint, we prove Theorem Theorem 1.1.1 as a corollary to a further generalization of the polynomial Carleson operator towards doubling metric measure spaces, Theorem Theorem 1.1.1.1 below, which is new and appears in the sibling communication. Theorem `metric-space-Carleson` is an axiomatic approach to Carleson type theorems on doubling metric measure spaces. This axiomatic approach is suitable for formalization and a good route towards the classical theorem.

Early drafts of the sibling communication existed in summer 2023. Based on this, the first draft of the present blueprint was written in the first half of 2024, containing a much more detailed proof, which involved increasing the size by a factor of four, and adding the derivation of Carleson's classical result. In June 2024, the fourth author launched a public website[1] to post the blueprint, using the open-source software `leanblueprint` developed by Patrick Massot, calling for contributions to formalize the proof. The goal was to formalize the blueprint in the Lean proof assistant, building on top of its mathematical library `Mathlib`. The work was split up into about 180 tasks, to be claimed by individual contributors. Most tasks were to formalize the proof of a single lemma from the blueprint, and some were to develop basic theory or refactor existing code. The contributors adapted the blueprint to fix some gaps found during the formalization and gave feedback that led to discussions about the proof. This even resulted in a few changes to the general setup and the main theorems. All of the gaps found required only fairly localized changes to the blueprint, indicating that the initial blueprint was already of high quality. The formalization was completed in July 2025. It is attached to this arXiv posting, and the latest version can be found on GitHub[2].

Everyone that completed a substantial amount of tasks is included as a coauthor of the blueprint. The authors acknowledge contributions in the form of small formalization additions, pointing out corrections to the blueprint, or supplying ideas to the Lean efforts by the following people: Michel Alexis, Bolton Bailey, Julian Berman, Joachim Breitner, Martin Dvorak, Georges Gonthier,

[1] `https://florisvandoorn.com/carleson/`
[2] `https://github.com/fpvandoorn/carleson`

Aaron Hill, Austin Letson, Bhavik Mehta, Eric Paul, Clara Torres, Dennis Tsar, Andrew Yang, Ruben van de Velde.

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Statement of the metric space Carleson theorems

Let

$$a \geq 4$$

be a natural number that as it gets larger will allow for more general applications of Theorem Theorem 1.1.1.1 but will worsen the constants in the estimates.

A doubling metric measure space (X, ρ, μ, a) is a complete and locally compact metric space (X, ρ) equipped with a non-zero locally finite Borel measure μ that satisfies the doubling condition that for all $x \in X$ and all $R > 0$ we have

$$\mu(B(x, 2R)) \leq 2^a \mu(B(x, R)),$$

where we have denoted by $B(x, R)$ the open ball of radius R centered at x:

$$B(x, R) := \{y \in X : \rho(x, y) < R\}.$$

In a doubling metric measure space the closed balls are compact and μ is positive on all non-empty open sets.

A collection Θ of real valued continuous functions on the doubling metric measure space (X, ρ, μ, a) is called compatible, if there is a point $o \in X$ where all the functions are equal to 0, and if there exists for each ball $B \subset X$ a metric d_B on Θ, such that the following five properties are satisfied. For every ball $B \subset X$

$$\sup_{x,y \in B} |\vartheta(x) - \vartheta(y) - \theta(x) + \theta(y)| \leq d_B(\vartheta, \theta).$$

For any two balls $B_1 = B(x_1, R)$, $B_2 = B(x_2, 2R)$ in X with $x_1 \in B_2$ and any $\vartheta, \theta \in \Theta$,

$$d_{B_2}(\vartheta, \theta) \leq 2^a d_{B_1}(\vartheta, \theta).$$

For any two balls B_1, B_2 in X with $B_1 \subset B_2$ and any $\vartheta, \theta \in \Theta$

$$d_{B_1}(\vartheta, \theta) \leq d_{B_2}(\vartheta, \theta)$$

and for any two balls $B_1 = B(x_1, R)$, $B_2 = B(x_2, 2^a R)$ with $B_1 \subset B_2$, and $\vartheta, \theta \in \Theta$,

$$2d_{B_1}(\vartheta, \theta) \leq d_{B_2}(\vartheta, \theta).$$

For every ball B in X and every d_B-ball $\tilde{B}$ of radius $2R$ in Θ, there is a collection $\mathcal{B}$ of at most 2^a many d_B-balls of radius R covering $\tilde{B}$, that is,

$$\tilde{B} \subset \bigcup \mathcal{B}.$$

Further, a compatible collection Θ is called cancellative, if for any ball B in X of radius R, any Lipschitz function $\varphi : X \to \mathbb{C}$ supported on B, and any $\vartheta, \theta \in \Theta$ we have

$$|\int_B e(\vartheta(x) - \theta(x))\varphi(x)\, d\mu(x)| \leq 2^a \mu(B) \|\varphi\|_{\mathrm{Lip}(B)} (1 + d_B(\vartheta, \theta))^{-1/a},$$

where $\|\cdot\|_{\mathrm{Lip}(B)}$ denotes the inhomogeneous Lipschitz norm on B:

$$\|\varphi\|_{\mathrm{Lip}(B)} = \sup_{x \in B} |\varphi(x)| + R \sup_{x,y \in B, x \neq y} \frac{|\varphi(x) - \varphi(y)|}{\rho(x,y)}.$$

A one-sided Calderon-Zygmund kernel K on the doubling metric measure space (X, ρ, μ, a) is a measurable function

$$K : X \times X \to \mathbb{C}$$

such that for all $x, y', y \in X$ with $x \neq y$, we have

$$|K(x,y)| \leq \frac{2^{a^3}}{V(x,y)}$$

and if $2\rho(y,y') \leq \rho(x,y)$, then

$$|K(x,y) - K(x,y')| \leq \left(\frac{\rho(y,y')}{\rho(x,y)}\right)^{1/a} \frac{2^{a^3}}{V(x,y)},$$

where $V(x,y) := \mu(B(x, \rho(x,y)))$. Define the maximally truncated non-tangential singular integral T_* associated with K by

$$T_* f(x) := \sup_{R_1 < R_2} \sup_{\rho(x,x') < R_1} \left| \int_{R_1 < \rho(x',y) < R_2} K(x',y) f(y)\, \mathrm{d}\mu(y) \right|.$$

We define the generalized Carleson operator T by

$$Tf(x) := \sup_{\vartheta \in \Theta} \sup_{0 < R_1 < R_2} \left| \int_{R_1 < \rho(x,y) < R_2} K(x,y) f(y) e(\vartheta(y))\, \mathrm{d}\mu(y) \right|,$$

where $e(r) = e^{ir}$.

Our first main result is the following restricted weak type estimate for T in the range $1 < q \leq 2$, which by interpolation techniques recovers L^q estimates for the open range $1 < q < 2$.

Theorem 1.1.1.1

For all integers $a \geq 4$ and real numbers $1 < q \leq 2$ the following holds. Let (X, ρ, μ, a) be a doubling metric measure space. Let Θ be a cancellative compatible collection of functions and let K be a one-sided Calderon-Zygmund kernel on (X, ρ, μ, a). Assume that for every bounded measurable function g on X supported on a set of finite measure we have

$$\|T_* g\|_2 \leq 2^{a^3} \|g\|_2,$$

where T_* is defined above. Then for all Borel sets F and G in X and all Borel functions $f : X \to \mathbb{C}$ with $|f| \leq \mathbf{1}_F$, we have, with T defined above,

$$\left| \int_G T f \, \mathrm{d}\mu \right| \leq \frac{2^{443a^3}}{(q-1)^6} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}}.$$

In some applications, such as the Walsh-case of Carleson's theorem, the kernel K naturally depends also on the modulation functions ϑ. The fact that we do not assume Holder-continuity of the kernel K in the first argument allows us to also capture this situation.

We now state another Theorem with a slightly weaker replacement of the assumption above. It implies the Walsh-case of Carleson's theorem. For a Borel function $Q : X \to \Theta$, and $\vartheta \in \Theta$ and $x \in X$ define

$$R_Q(\vartheta, x) = \sup\{r > 0 : d_{B(x,r)}(\vartheta, Q(x)) < 1\}$$

and define further

$$T_Q^{\vartheta} f(x) := \sup_{R_1 < R_2} \sup_{\rho(x,x') < R_1} \left| \int_{R_1 < \rho(x',y) < \min\{R_2, R_Q(\vartheta,x')\}} K(x', y) f(y) \, \mathrm{d}\mu(y) \right|$$

Define further the linearized generalized Carleson operator T_Q by

$$T_Q f(x) := \sup_{0 < R_1 < R_2} \left| \int_{R_1 < \rho(x,y) < R_2} K(x, y) f(y) e(Q(x)(y)) \, \mathrm{d}\mu(y) \right|,$$

where again $e(r) = e^{ir}$.

Theorem 1.1.1.2

For all integers $a \geq 4$ and real numbers $1 < q \leq 2$ the following holds. Let (X, ρ, μ, a) be a doubling metric measure space. Let Θ be a cancellative compatible collection of functions. Let $Q : X \to \Theta$ be a Borel function with finite range. Let K be a one-sided Calderon-Zygmund kernel on (X, ρ, μ, a). Assume that for every $\vartheta \in \Theta$ and every bounded measurable function g on X supported on a set of finite measure we have

$$\|T_Q^{\vartheta} g\|_2 \leq 2^{a^3} \|g\|_2,$$

where T_Q^ϑ is defined above. Then for all Borel sets F and G in X and all Borel functions $f : X \to \mathbb{C}$ with $|f| \le \mathbf{1}_F$, we have, with T_Q defined above,

$$\left|\int_G T_Q f \,\mathrm{d}\mu\right| \le \frac{2^{443a^3}}{(q-1)^6}\mu(G)^{1-\frac{1}{q}}\mu(F)^{\frac{1}{q}}.$$

The value of the constant factor 2^{443a^3} in Theorems Theorem 1.1.1.1 and Theorem 1.1.1.2 is by no means sharp. One source of non-sharpness is our choice to write for readability most constants in the form 2^{na^3} for some explicit constant n.

An a posteriori byproduct of the Lean formalization of this document is that Theorems Theorem 1.1.1.1 and Theorem 1.1.1.2 remain true with 2^{44a^3} instead of 2^{443a^3}. This is obtained by changing the parameter D introduced in `defineD` from 2^{100a^2} to 2^{7a^2}, checking that exactly the same proof goes through, tracking the constants, and getting 2^{44a^3} in the end. This value is again by no means sharp. In the formalization we define the parameter D as 2^{a^2} and the constants in this blueprint are obtained by setting $= 100$.

1.2 Proof of Metric Space Carleson, overview

This section organizes the proof of Theorem 1.1.1.1 into sections section 1.3 (p. 17), section 1.4 (p. 23), section 1.5 (p. 36), section 1.6 (p. 59), section 1.7 (p. 75), section 1.8 (p. 117), and section 1.9 (p. 120). These sections are mutually independent except for referring to the statements formulated in the present section. section 1.3 (p. 17) proves the main Theorem 1.1.1.1, while sections section 1.4 (p. 23), section 1.5 (p. 36), section 1.6 (p. 59), section 1.7 (p. 75), section 1.8 (p. 117), and section 1.9 (p. 120) each prove one proposition that is stated in the present section. The present section also introduces all definitions used across these sections.

section 1.2 (p. 14) proves some auxiliary lemmas that are used in more than one of the sections 3-9.

Let a, q be given as in Theorem 1.1.1.1.

Define

$$D := 2^{100a^2},$$

$$\kappa := 2^{-10a},$$

and

$$Z := 2^{12a}.$$

Let $\psi : \mathbb{R} \to \mathbb{R}$ be the unique compactly supported, piece-wise linear, continuous function with corners precisely at $\frac{1}{4D}$, $\frac{1}{2D}$, $\frac{1}{4}$ and $\frac{1}{2}$ which satisfies

$$\sum_{s\in\mathbb{Z}} \psi(D^{-s}x) = 1$$

for all $x > 0$. This function vanishes outside $[\frac{1}{4D}, \frac{1}{2}]$, is constant one on $[\frac{1}{2D}, \frac{1}{4}]$, and is Lipschitz with constant $4D$.

Let a doubling metric measure space (X, ρ, μ, a) be given. Let a cancellative compatible collection Θ of functions on X be given. Let $o \in X$ be a point such that $\vartheta(o) = 0$ for all $\vartheta \in \Theta$.

Let a Borel measurable function $Q : X \to \Theta$ with finite range be given. Let a one-sided Calderon-Zygmund kernel K on X be given so that for every $\vartheta \in \Theta$ the operator T_Q^ϑ defined above satisfies the assumption `linnontanbound`.

For $s \in \mathbb{Z}$, we define

$$K_s(x, y) := K(x, y)\psi(D^{-s}\rho(x, y)),$$

so that for each $x, y \in X$ with $x \neq y$ we have

$$K(x, y) = \sum_{s \in \mathbb{Z}} K_s(x, y).$$

In section 1.3 (p. 17), we prove Theorem 1.1.1.1 and Theorem 1.1.1.2 from the finitary version, Theorem 1.2.1 below. Recall that a function from a measure space to a finite set is measurable if the pre-image of each of the elements in the range is measurable.

Theorem 1.2.1

> Let $\sigma_1, \sigma_2 \colon X \to \mathbb{Z}$ be measurable functions with finite range and $\sigma_1 \leq \sigma_2$. Let F, G be bounded Borel sets in X. Then there is a Borel set G' in X with $2\mu(G') \leq \mu(G)$ such that for all Borel functions $f : X \to \mathbb{C}$ with $|f| \leq \mathbf{1}_F$,

$$\int_{G \setminus G'} \left| \sum_{s=\sigma_1(x)}^{\sigma_2(x)} \int K_s(x, y) f(y) e(Q(x)(y)) \, \mathrm{d}\mu(y) \right| \mathrm{d}\mu(x) \leq \frac{2^{442a^3}}{(q-1)^5} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}}.$$

Let measurable functions $\sigma_1 \leq \sigma_2 \colon X \to \mathbb{Z}$ with finite range be given. Let bounded Borel sets F, G in X be given. Let S be the smallest integer such that the ranges of σ_1 and σ_2 are contained in $[-S, S]$ and F and G are contained in the ball $B(o, \frac{1}{4}D^S)$.

In section 1.4 (p. 23), we prove Theorem 1.2.1 using a bound for a dyadic model formulated in Theorem 1.2.2 below.

A grid structure $(\mathcal{D}, c, s)$ on X consists of a finite collection $\mathcal{D}$ of pairs (I, k) of Borel sets in X and integers $k \in [-S, S]$, the projection $s \colon \mathcal{D} \to [-S, S], (I, k) \mapsto k$ to the second component which is assumed to be surjective and called scale function, and a function $c : \mathcal{D} \to X$ called center function such that the five properties `coverdyadic`, `dyadicproperty`, `subsetmaxcube`, `eq-vol-sp-cube`, and `eq-small-boundary` hold. We call the elements of $\mathcal{D}$ dyadic cubes. By abuse of notation, we will usually write just I for the cube (I, k), and we will write $I \subset J$ to mean that for two cubes $(I, k), (J, l) \in \mathcal{D}$ we have $I \subset J$ and $k \leq l$.

For each dyadic cube I and each $-S \le k < s(I)$ we have

$$I \subset \bigcup_{J \in \mathcal{D}: s(J)=k} J.$$

Any two non-disjoint dyadic cubes I, J with $s(I) \le s(J)$ satisfy

$$I \subset J.$$

There exists a $I_0 \in \mathcal{D}$ with $s(I_0) = S$ and $c(I_0) = o$ and for all cubes $J \in \mathcal{D}$, we have

$$J \subset I_0.$$

For any dyadic cube I,

$$c(I) \in B(c(I), \frac{1}{4}D^{s(I)}) \subset I \subset B(c(I), 4D^{s(I)}).$$

For any dyadic cube I and any t with $tD^{s(I)} \ge D^{-S}$,

$$\mu(\{x \in I \ : \ \rho(x, X \setminus I) \le tD^{s(I)}\}) \le 2t^\kappa \mu(I).$$

A tile structure $(\mathfrak{P}, \mathcal{I}, \Omega, \mathcal{Q}, \mathrm{c}, \mathrm{s})$ for a given grid structure $(\mathcal{D}, c, s)$ is a finite set $\mathfrak{P}$ of elements called tiles with five maps

$$\mathcal{I} \colon \mathfrak{P} \to \mathcal{D}$$

$$\Omega \colon \mathfrak{P} \to \mathcal{P}(\Theta)$$

$$\mathcal{Q} \colon \mathfrak{P} \to Q(X)$$

$$\mathrm{c} \colon \mathfrak{P} \to X$$

$$\mathrm{s} \colon \mathfrak{P} \to \mathbb{Z}$$

with $\mathcal{I}$ surjective and $\mathcal{P}(\Theta)$ denoting the power set of Θ such that the five properties `eq-dis-freq-cover`, `eq-freq-dyadic`, `eq-freq-comp-ball`, `tilecenter`, and `tilescale` hold. For each dyadic cube I, the restriction of the map Ω to the set

$$\mathfrak{P}(I) = \{\mathfrak{p} : \mathcal{I}(\mathfrak{p}) = I\}$$

is injective and we have the disjoint covering property

$$Q(X) \subset \dot{\bigcup}_{\mathfrak{p} \in \mathfrak{P}(I)} \Omega(\mathfrak{p}).$$

For any tiles $\mathfrak{p}, \mathfrak{q}$ with $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{q})$ and $\Omega(\mathfrak{p}) \cap \Omega(\mathfrak{q}) \ne \emptyset$ we have

$$\Omega(\mathfrak{q}) \subset \Omega(\mathfrak{p}).$$

For each tile $\mathfrak{p}$,

$$\mathcal{Q}(\mathfrak{p}) \in B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 0.2) \subset \Omega(\mathfrak{p}) \subset B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 1),$$

where

$$B_{\mathfrak{p}}(\vartheta, R) := \{\theta \in \Theta \,:\, d_{\mathfrak{p}}(\vartheta, \theta) < R\},$$

and

$$d_{\mathfrak{p}} := d_{B(\mathrm{c}(\mathfrak{p}), \frac{1}{4} D^{\mathrm{s}(\mathfrak{p})})}.$$

We have for each tile $\mathfrak{p}$

$$\mathrm{c}(\mathfrak{p}) = c(\mathcal{I}(\mathfrak{p})),$$

$$\mathrm{s}(\mathfrak{p}) = s(\mathcal{I}(\mathfrak{p})).$$

Theorem 1.2.2

Let $(\mathcal{D}, c, s)$ be a grid structure and

$$(\mathfrak{P}, \mathcal{I}, \Omega, \mathcal{Q}, \mathrm{c}, \mathrm{s})$$

a tile structure for this grid structure. Define for $\mathfrak{p} \in \mathfrak{P}$

$$E(\mathfrak{p}) = \{x \in \mathcal{I}(\mathfrak{p}) : Q(x) \in \Omega(\mathfrak{p}), \sigma_1(x) \le \mathrm{s}(\mathfrak{p}) \le \sigma_2(x)\}$$

and

$$T_{\mathfrak{p}} f(x) = \mathbf{1}_{E(\mathfrak{p})}(x) \int K_{\mathrm{s}(\mathfrak{p})}(x, y) f(y) e(Q(x)(y) - Q(x)(x)) \, d\mu(y).$$

Then there exists a Borel set G' with $2\mu(G') \le \mu(G)$ such that for all Borel functions $f : X \to \mathbb{C}$ with $|f| \le \mathbf{1}_F$ we have

$$\int_{G \setminus G'} \left| \sum_{\mathfrak{p} \in \mathfrak{P}} T_{\mathfrak{p}} f(x) \right| \mathrm{d}\mu(x) \le \frac{2^{442a^3}}{(q-1)^5} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}}.$$

The proof of Theorem 1.2.2 is done in section 1.5 (p. 36) by a reduction to two further propositions that we state below.

Fix a grid structure $(\mathcal{D}, c, s)$ and a tile structure $(\mathfrak{P}, \mathcal{I}, \Omega, \mathcal{Q}, \mathrm{c}, \mathrm{s})$ for this grid structure.

We define the relation

$$\mathfrak{p} \le \mathfrak{p}'$$

on $\mathfrak{P} \times \mathfrak{P}$ meaning $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{p}')$ and $\Omega(\mathfrak{p}') \subset \Omega(\mathfrak{p})$. We further define for $\lambda, \lambda' > 0$ the relation

$$\lambda \mathfrak{p} \lesssim \lambda' \mathfrak{p}'$$

on $\mathfrak{P} \times \mathfrak{P}$ meaning $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{p}')$ and

$$B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \lambda') \subset B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \lambda).$$

Define for a tile $\mathfrak{p}$ and $\lambda > 0$

$$E_1(\mathfrak{p}) := \{x \in \mathcal{I}(\mathfrak{p}) \cap G : Q(x) \in \Omega(\mathfrak{p})\},$$

$$E_2(\lambda, \mathfrak{p}) := \{x \in \mathcal{I}(\mathfrak{p}) \cap G : Q(x) \in B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \lambda)\}.$$

Given a subset $\mathfrak{P}'$ of $\mathfrak{P}$, we define $\mathfrak{P}(\mathfrak{P}')$ to be the set of all $\mathfrak{p} \in \mathfrak{P}$ such that there exist $\mathfrak{p}' \in \mathfrak{P}'$ with $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{p}')$. Define the densities

$$\mathrm{dens}_1(\mathfrak{P}') := \sup_{\mathfrak{p}' \in \mathfrak{P}'} \sup_{\lambda \geq 2} \lambda^{-a} \sup_{\mathfrak{p} \in \mathfrak{P}(\mathfrak{P}'), \lambda\mathfrak{p}' \lesssim \lambda\mathfrak{p}} \frac{\mu(E_2(\lambda, \mathfrak{p}))}{\mu(\mathcal{I}(\mathfrak{p}))},$$

$$\mathrm{dens}_2(\mathfrak{P}') := \sup_{\mathfrak{p}' \in \mathfrak{P}'} \sup_{r \geq 4D^{\mathrm{s}(\mathfrak{p})}} \frac{\mu(F \cap B(\mathrm{c}(\mathfrak{p}), r))}{\mu(B(\mathrm{c}(\mathfrak{p}), r))}.$$

An antichain is a subset $\mathfrak{A}$ of $\mathfrak{P}$ such that for any distinct $\mathfrak{p}, \mathfrak{q} \in \mathfrak{A}$ we do not have $\mathfrak{p} \leq \mathfrak{q}$.

The following proposition is proved in section 1.6 (p. 59).

Theorem 1.2.3

For any antichain $\mathfrak{A}$ and for all $f : X \to \mathbb{C}$ with $|f| \leq \mathbf{1}_F$ and all $g : X \to \mathbb{C}$ with $|g| \leq \mathbf{1}_G$

$$|\int \overline{g(x)} \sum_{\mathfrak{p} \in \mathfrak{A}} T_{\mathfrak{p}} f(x)\, d\mu(x)| \leq \frac{2^{117a^3}}{q-1} \mathrm{dens}_1(\mathfrak{A})^{\frac{q-1}{8a^4}} \mathrm{dens}_2(\mathfrak{A})^{\frac{1}{q}-\frac{1}{2}} \|f\|_2 \|g\|_2.$$

Let $n \geq 0$. An n-forest is a pair $(\mathfrak{U}, \mathfrak{T})$ where $\mathfrak{U}$ is a subset of $\mathfrak{P}$ and $\mathfrak{T}$ is a map assigning to each $\mathfrak{u} \in \mathfrak{U}$ a nonempty set $\mathfrak{T}(\mathfrak{u}) \subset \mathfrak{P}$ called tree such that the following properties `forest1`, `forest2`, `forest3`, `forest4`, `forest5`, and `forest6` hold.

For each $\mathfrak{u} \in \mathfrak{U}$ and each $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ we have $\mathcal{I}(\mathfrak{p}) \neq \mathcal{I}(\mathfrak{u})$ and

$$4\mathfrak{p} \lesssim \mathfrak{u}.$$

For each $\mathfrak{u} \in \mathfrak{U}$ and each $\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{T}(\mathfrak{u})$ and $\mathfrak{p}' \in \mathfrak{P}$ we have

$$\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{T}(\mathfrak{u}), \mathfrak{p} \leq \mathfrak{p}' \leq \mathfrak{p}'' \implies \mathfrak{p}' \in \mathfrak{T}(\mathfrak{u}).$$

We have

$$\|\sum_{\mathfrak{u} \in \mathfrak{U}} \mathbf{1}_{\mathcal{I}(\mathfrak{u})}\|_\infty \leq 2^n.$$

We have for every $\mathfrak{u} \in \mathfrak{U}$

$$\mathrm{dens}_1(\mathfrak{T}(\mathfrak{u})) \leq 2^{4a+1-n}.$$

We have for $\mathfrak{u}, \mathfrak{u}' \in \mathfrak{U}$ with $\mathfrak{u} \neq \mathfrak{u}'$ and $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}')$ with $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{u})$ that

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{u})) > 2^{Z(n+1)}.$$

We have for every $\mathfrak{u} \in \mathfrak{U}$ and $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ that

$$B(\mathrm{c}(\mathfrak{p}), 8D^{\mathrm{s}(\mathfrak{p})}) \subset \mathcal{I}(\mathfrak{u}).$$

The following proposition is proved in section 1.7 (p. 75).

Theorem 1.2.4

For any $n \geq 0$ and any n-forest $(\mathfrak{U}, \mathfrak{T})$ we have for all $f, g : X \to \mathbb{C}$ with $|f| \leq \mathbf{1}_F$ and $|g| \leq \mathbf{1}_G$

$$|\int \overline{g(x)} \sum_{\mathfrak{u} \in \mathfrak{U}} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f(x) \, \mathrm{d}\mu(x)|$$

$$\leq 2^{440a^3} 2^{-\frac{q-1}{q}n} \operatorname{dens}_2 \left(\bigcup_{\mathfrak{u} \in \mathfrak{U}} \mathfrak{T}(\mathfrak{u}) \right)^{\frac{1}{q} - \frac{1}{2}} \|f\|_2 \|g\|_2.$$

Theorem 1.1.1.1 is formulated at the level of generality for general kernels satisfying the mere Holder regularity condition `eqkernel-y-smooth`. On the other hand, the cancellative condition `eq-vdc-cond` is a testing condition against more regular, namely Lipschitz functions. To bridge the gap, we follow the literature to observe a variant of `eq-vdc-cond` that we formulate in the following proposition proved in section 1.8 (p. 117).

Define

$$\tau := \frac{1}{a}.$$

Define for any open ball B of radius R in X the L^∞-normalized τ-Holder norm by

$$\|\varphi\|_{C^\tau(B)} = \sup_{x \in B} |\varphi(x)| + R^\tau \sup_{x,y \in B, x \neq y} \frac{|\varphi(x) - \varphi(y)|}{\rho(x,y)^\tau}.$$

Theorem 1.2.5

Let $z \in X$ and $R > 0$ and set $B = B(z, R)$. Let $\varphi : X \to \mathbb{C}$ be supported on B and satisfy $\|\varphi\|_{C^\tau(B(z,2R))} < \infty$. Let $\vartheta, \theta \in \Theta$. Then

$$|\int e(\vartheta(x) - \theta(x)) \varphi(x) dx| \leq 2^{7a} \mu(B) \|\varphi\|_{C^\tau(B(z,2R))} (1 + d_B(\vartheta, \theta))^{-\frac{1}{2a^2 + a^3}}.$$

We further formulate a classical Vitali covering result and maximal function estimate that we need throughout several sections. This following proposition will typically be applied to the absolute value of a complex valued function and be proved in section 1.9 (p. 120). By a ball B we mean a set $B(x, r)$ with $x \in X$ and $r > 0$ as defined in `eq-define-ball`. For a finite collection $\mathcal{B}$ of balls in X and $1 \leq p < \infty$ define the measurable function $M_{\mathcal{B},p} u$ on X by

$$M_{\mathcal{B},p} u(x) := \left(\sup_{B \in \mathcal{B}} \frac{\mathbf{1}_B(x)}{\mu(B)} \int_B |u(y)|^p \, d\mu(y) \right)^{\frac{1}{p}},$$

Define further $M_{\mathcal{B}} := M_{\mathcal{B},1}$.
Theorem 1.2.6

Let $\mathcal{B}$ be a finite collection of balls in X. If for some $\lambda > 0$ and some measurable function $u : X \to [0, \infty)$ we have

$$\int_B u(x)\, d\mu(x) \geq \lambda\mu(B)$$

for each $B \in \mathcal{B}$, then

$$\lambda\mu(\bigcup \mathcal{B}) \leq 2^{2a} \int_X u(x)\, d\mu(x).$$

For every measurable function v and $1 \leq p_1 < p_2$ we have

$$\|M_{\mathcal{B},p_1} v\|_{p_2} \leq 2^{2a} \frac{p_2}{p_2 - p_1} \|v\|_{p_2}.$$

Moreover, given any measurable bounded function $w : X \to \mathbb{C}$ there exists a measurable function $Mw : X \to [0, \infty)$ such that the following hold. For each ball $B \subset X$ and each $x \in B$

$$\frac{1}{\mu(B)} \int_B |w(y)|\, \mathrm{d}\mu(y) \leq Mw(x)$$

and for all $1 \leq p_1 < p_2 \leq \infty$

$$\|M(w^{p_1})^{\frac{1}{p_1}}\|_{p_2} \leq 2^{4a} \frac{p_2}{p_2 - p_1} \|w\|_{p_2}.$$

This completes the overview of the proof of Theorem 1.1.1.1.

Auxiliary lemmas

We close this section by recording some auxiliary lemmas about the objects defined in section 1.2 (p. 8), which will be used in multiple sections to follow.

First, we record an estimate for the metrical entropy numbers of balls in the space Θ equipped with any of the metrics d_B, following from the doubling property `thirddb`.

Lemma 1.2.1.1

Let $B' \subset X$ be a ball. Let $r > 0$, $\vartheta \in \Theta$ and $k \in \mathbb{N}$. Suppose that $\mathcal{Z} \subset B_{B'}(\vartheta, r2^k)$ satisfies that $\{B_{B'}(z, r) \mid z \in \mathcal{Z}\}$ is a collection of pairwise disjoint sets. Then

$$|\mathcal{Z}| \leq 2^{ka}.$$

Proof for Lemma 1.2.1.1

Proof. By applying property `thirddb` k times, we obtain a collection $\mathcal{Z}' \subset \Theta$ with $|\mathcal{Z}'| = 2^{ka}$ and

$$B_{B'}(\vartheta, r2^k) \subset \bigcup_{z' \in \mathcal{Z}'} B_{B'}(z', \frac{r}{2}).$$

Then each $z \in \mathcal{Z}$ is contained in one of the balls $B(z', \frac{r}{2})$, but by the separation assumption no such ball contains more than one element of $\mathcal{Z}$. Thus $|\mathcal{Z}| \leq |\mathcal{Z}'| = 2^{ka}$.

The next lemma concerns monotonicity of the metrics $d_{B(c(I), \frac{1}{4}D^{s(I)})}$ with respect to inclusion of cubes I in a grid.

Lemma 1.2.1.2

Let $(\mathcal{D}, c, s)$ be a grid structure. Denote for cubes $I \in \mathcal{D}$

$$I^\circ := B(c(I), \frac{1}{4}D^{s(I)}).$$

Let $I, J \in \mathcal{D}$ with $I \subset J$. Then for all $\vartheta, \theta \in \Theta$ we have

$$d_{I^\circ}(\vartheta, \theta) \leq d_{J^\circ}(\vartheta, \theta),$$

and if $I \neq J$ then we have

$$d_{I^\circ}(\vartheta, \theta) \leq 2^{-95a} d_{J^\circ}(\vartheta, \theta).$$

Proof for Lemma 1.2.1.2

Proof. If $s(I) \geq s(J)$ then `dyadicproperty` and the assumption $I \subset J$ imply $I = J$. Then the lemma holds by reflexivity. If $s(J) \geq s(I)+1$, then using the monotonicity property `monotonedb`, `defineD` and `seconddb`, we get

$$d_{I^\circ}(\vartheta, \theta) \leq d_{B(c(I), 4D^{s(I)})}(\vartheta, \theta) \leq 2^{-100a} d_{B(c(I), 4D^{s(J)})}(\vartheta, \theta).$$

Using `eq-vol-sp-cube`, together with the inclusion $I \subset J$, we obtain

$$c(I) \in I \subset J \subset B(c(J), 4D^{s(J)})$$

and consequently by the triangle inequality

$$B(c(I), 4D^{s(J)}) \subset B(c(J), 8D^{s(J)}).$$

Using this together with the monotonicity property `monotonedb` and `firstdb` in the previous estimate, we obtain

$$\begin{aligned} d_{I^\circ}(\vartheta, \theta) &\leq 2^{-100a} d_{B(c(J), 8D^{s(J)})}(\vartheta, \theta) \\ &\leq 2^{-100a+5a} d_{B(c(J), \frac{1}{4}D^{s(J)})}(\vartheta, \theta) \\ &= 2^{-95a} d_{J^\circ}(\vartheta, \theta). \end{aligned}$$

This proves the second inequality claimed in the Lemma, from which the first follows since $a \geq 4$ and hence $2^{-95a} \leq 1$.

We also record the following basic estimates for the kernels K_s.

Lemma 1.2.1.3

Let $-S \leq s \leq S$ and $x, y, y' \in X$. If $K_s(x,y) \neq 0$, then we have

$$\frac{1}{4}D^{s-1} \leq \rho(x,y) \leq \frac{1}{2}D^s.$$

We have

$$|K_s(x,y)| \leq \frac{2^{102a^3}}{\mu(B(x,D^s))},$$

and

$$|K_s(x,y) - K_s(x,y')| \leq \frac{2^{127a^3}}{\mu(B(x,D^s))}\left(\frac{\rho(y,y')}{D^s}\right)^{\frac{1}{a}}.$$

Proof for Lemma 1.2.1.3

Proof. By Definition `defks`, the function K_s is the product of K with a function which is supported in the set of all x, y satisfying `supp-Ks`. This proves `supp-Ks`. Using `eqkernel-size` and the lower bound in `supp-Ks` we obtain

$$|K_s(x,y)| \leq |K(x,y)| \leq \frac{2^{a^3}}{\mu(B(x,\frac{1}{4}D^{s-1}))}.$$

Using $D = 2^{100a^2}$ and the doubling property `doublingx` $2 + 100a^2$ times estimates the last display by

$$\leq \frac{2^{2a+101a^3}}{\mu(B(x,D^s))}.$$

Using $a \geq 4$ proves `eq-Ks-size`. To prove `eq-Ks-smooth` when $2\rho(y,y') > \rho(x,y)$, use the lower bound in `supp-Ks` and the inequality $2\rho(y,y') > \frac{1}{4}D^{s-1}$. Then `eq-Ks-smooth` follows from the triangle inequality, `eq-Ks-size` and $a \geq 4$. If $2\rho(y,y') \leq \rho(x,y)$, we rewrite $|K_s(x,y) - K_s(x,y')|$ as

$$|(K(x,y)-K(x,y'))\psi(D^{-s}\rho(x,y))+K(x,y)(\psi(D^{-s}\rho(x,y))-\psi(D^{-s}\rho(x,y')))|.$$

An upper bound for $|K(x,y) - K(x,y')|$ is obtained similarly to the proof of `eq-Ks-size`, using `eqkernel-y-smooth` and the lower bound in `supp-Ks`

$$|K(x,y) - K(x,y')| \leq \frac{2^{a^3}}{\mu(B(x,\frac{1}{4}D^{s-1}))}\left(\frac{\rho(y,y')}{\frac{1}{4}D^{s-1}}\right)^{\frac{1}{a}}.$$

As above, this is estimated by

$$\leq \frac{4D2^{2a+101a^3}}{\mu(B(x,D^s))}\left(\frac{\rho(y,y')}{D^s}\right)^{\frac{1}{a}} = \frac{2^{2+2a+100a^2+101a^3}}{\mu(B(x,D^s))}\left(\frac{\rho(y,y')}{D^s}\right)^{\frac{1}{a}}.$$

We have the trivial bound $|\psi(D^{-s}\rho(x,y))| \leq 1$, and `eq-Ks-aux` provides a bound for $|K(x,y)|$. Finally, we show that

$$|\psi(D^{-s}\rho(x,y)) - \psi(D^{-s}\rho(x,y'))| \leq 4D\left(\frac{\rho(y,y')}{D^s}\right)^{\frac{1}{a}}$$

by considering separately the cases $\rho(y,y')/D^s \geq 1$ and $\rho(y,y')/D^s < 1$. In the former case, the inequality is trivial; in the latter case, it follows from the fact that ψ is Lipschitz with constant $4D$. Combining the above bounds and using $a \geq 4$ proves `eq-Ks-smooth` when $2\rho(y,y') \leq \rho(x,y)$.

1.3 Proof of Metric Space Carleson

In this section we prove Theorem 1.1.1.1 and Theorem 1.1.1.2.

Let (X,ρ,μ,a) be a doubling metric measure space and Θ a cancellative, compatible collection of functions on X. Let K be a one-sided Calderon-Zygmund kernel.

We begin by proving some continuity properties of the integrand in `def-main-op`.

Lemma 1.3.1

Let f be a measurable function with $|f| \leq 1$. Then the function

$$G : X \times \Theta \times (0,\infty) \times (0,\infty) \to \mathbb{C}$$

$$G(x,\vartheta,R_1,R_2) := \int_{R_1<\rho(x,y)<R_2} K(x,y)f(y)e(\vartheta(y))\,\mathrm{d}\mu(y)$$

is continuous in ϑ for fixed x, R_1, R_2. It is right-continuous in R_1 for fixed x, ϑ, R_2 and left-continuous in R_2 for fixed x, ϑ, R_1. Finally, it is measurable in x and bounded for fixed ϑ, R_1, R_2.

Proof for Lemma 1.3.1

Proof. Measurability in x follows from joint measurability of

$$K(x,y)\mathbf{1}_{B(x,R_2)\setminus B(x,R_1)}(y)$$

in x and y and part of the proof of Fubini's theorem. Partial continuity in R_1 and R_2 is also a standard lemma in measure theory. It follows for example by splitting the integrand as $F_1 - F_{-1} + iF_i - iF_{-i}$ for positive, disjointly supported functions F_- and applying the monotone convergence theorem to each summand. For continuity in ϑ note that

$$|G(x,\vartheta,R_1,R_2)-G(x,\vartheta',R_1,R_2)|$$

$$=\left|\int_{R_1<\rho(x,y)<R_2} K(x,y)f(y)(e(\vartheta(y))-e(\vartheta'(y)))\,\mathrm{d}\mu(y)\right|$$

$$\leq \int_{R_1<\rho(x,y)<R_2} |K(x,y)||f(y)||e(\vartheta(y)-\vartheta'(y)-\vartheta(o)+\vartheta'(o))-1|\,\mathrm{d}\mu(y).$$

By 1-Lipschitz continuity of e^{ix}, the property `osccontrol` of the metrics d, the kernel upper bound `eqkernel-size` and $|f|\leq 1$ this is

$$\leq \mu(B(x,R_2)) \sup_{R_1<\rho(x,y)<R_2} \frac{2^{a^3}}{V(x,y)} d_{B(x,\rho(o,x)+R_2)}(\vartheta,\vartheta').$$

If $R_1<\rho(x,y)<R_2$ then there exists n with $B(x,R_2)\subset B(x,2^n\rho(x,y))$ and $2^n\leq 2R_2/R_1$. Hence, by the doubling property `doublingx`,

$$V(x,y)=\mu(B(x,\rho(x,y)))\geq 2^{-an}\mu(B(x,R_2))\geq (2R_2/R_1)^{-a}\mu(B(x,R_2)).$$

Hence

$$|G(x,\vartheta,R_1,R_2)-G(x,\vartheta',R_1,R_2)|\leq 2^{a^3}\left(\frac{2R_2}{R_1}\right)^a d_{B(x,\rho(o,x)+R_2)}(\vartheta,\vartheta').$$

Since the topology on Θ is the one induced by any of the metrics d_B, continuity follows. Finally, for boundedness as a function of x note that we also have by a similar computation using $|e(\vartheta)|=1$

$$|G(x,\vartheta,R_1,R_2)|\leq 2^{a^3}\left(\frac{2R_2}{R_1}\right)^a.$$

We now prove Theorem Theorem 1.1.1.1 using Theorem Theorem 1.1.1.2.

Proof for Theorem 1.1.1.1

Proof of Theorem 1.1.1.1. Let Borel sets F, G in X be given. Let a Borel function $f: X\to\mathbb{C}$ with $f\leq \mathbf{1}_F$ be given. Let $\Theta'\subset\Theta$ be a countable dense set. By Lemma Lemma 1.3.1 we have

$$\sup_{\vartheta\in\Theta}\sup_{0<R_1<R_2}\left|\int_{R_1<\rho(x,y)<R_2} K(x,y)f(y)e(\vartheta(y))\,\mathrm{d}\mu(y)\right|$$

$$=\sup_{\vartheta\in\Theta'}\sup_{0<R_1<R_2, R_i\in\mathbb{Q}}\left|\int_{R_1<\rho(x,y)<R_2} K(x,y)f(y)e(\vartheta(y))\,\mathrm{d}\mu(y)\right|.$$

Consider an enumeration of Θ' and let Θ_n be the finite set consisting of the first $n+1$ functions in the enumeration. Then by the monotone convergence theorem

$$\Big|\int_G Tf\,\mathrm{d}\mu\Big|$$

$$= \int_G \sup_{\vartheta\in\Theta'} \sup_{0<R_1<R_2,R_i\in\mathbb{Q}} \left|\int_{R_1<\rho(x,y)<R_2} K(x,y)f(y)e(\vartheta(y))\,\mathrm{d}\mu(y)\right| \mathrm{d}\mu(x)$$

$$= \lim_{n\to\infty} \int_G \sup_{\vartheta\in\Theta_n} \sup_{0<R_1<R_2,R_i\in\mathbb{Q}} \left|\int_{R_1<\rho(x,y)<R_2} K(x,y)f(y)e(\vartheta(y))\,\mathrm{d}\mu(y)\right| \mathrm{d}\mu(x).$$

For each n, let $Q_n(x)$ be the measurable function specifying the maximizer in the supremum in ϑ in the previous line. It is possible to construct such a measurable function of x, because the function inside the supremum is measurable in x, being a countable supremum of measurable functions, and because Θ_n is finite. Then the previous display becomes

$$\leq \lim_{n\to\infty} \int_G \sup_{0<R_1<R_2} \left|\int_{R_1<\rho(x,y)<R_2} K(x,y)f(y)e(Q_n(x)(y))\,\mathrm{d}\mu(y)\right| \mathrm{d}\mu(x)$$

$$= \lim_{n\to\infty} \int_G T_{Q_n} f\,\mathrm{d}\mu.$$

It remains to verify the assumptions of Theorem Theorem 1.1.1.2, which when applied here completes the proof. The assumptions of Theorem Theorem 1.1.1.1 and Theorem Theorem 1.1.1.2 are the same, with the exception of the assumption `linnontanbound`. The assumption `linnontanbound` however is weaker than the assumption `nontanbound` of Theorem Theorem 1.1.1.1. Indeed, setting for fixed x,ϑ the outer radius R_2 in `def-tang-unm-op` to $\min\{R_2', R_Q(\vartheta,x')\}$, where R_2' is the outer radius in `def-lin-star-op`, shows that `def-lin-star-op` is smaller than or equal to `def-tang-unm-op`. Thus we can apply Theorem Theorem 1.1.1.2, which completes the proof.

We continue with the proof of Theorem Theorem 1.1.1.2, via a series of reductions to simpler lemmas.

Let a measurable function Q with finite range be given.

Proof for Theorem 1.1.1.2

Proof of Theorem 1.1.1.2. Let Borel sets F, G in X with finite measure be given. Let a Borel function $f : X \to \mathbb{C}$ with $f \leq \mathbf{1}_F$ be given. For each $0 < R_1, R_2, R$, we define $T_{R_1,R_2,R} f$ as in TRR. By

Lemma Lemma 1.3.1, $T_{R_1,R_2,R}f$ is measurable and bounded, and we clearly have for each $x \in X$

$$T_Q f(x) = \lim_{n\to\infty} \sup_{2^{-n}<R_1<R_2<2^n} T_{R_1,R_2,2^n} f(x).$$

For each x and all f, the functions $\sup_{2^{-n}<R_1<R_2<2^n} T_{R_1,R_2,2^n} f(x)$ are measurable by Lemma Lemma 1.3.1 and form an increasing sequence in n. By the monotone convergence theorem, the claimed estimate `linresweak` then follows from Lemma Lemma 1.3.2.

Lemma 1.3.2

Let F, G be Borel sets in X. Let $f : X \to \mathbb{C}$ be a Borel function with $|f| \le 1_F$. Then for all $R \in 2^{\mathbb{N}}$ we have

$$\int \mathbf{1}_G \sup_{1/R<R_1<R_2<R} |T_{R_1,R_2,R} f(x)|\, d\mu(x) \le \frac{2^{443a^3}}{(q-1)^6} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}},$$

where

$$T_{R_1,R_2,R} f(x) = \mathbf{1}_{B(o,R)}(x) \int_{R_1<\rho(x,y)<R_2} K(x,y) f(y) e(Q(x)(y))\, \mathrm{d}\mu(y).$$

Proof for Lemma 1.3.2

Proof. Let F, G, f as in the lemma be given. Let $R \in 2^{\mathbb{N}}$ be given. By replacing G with $G \cap B(o,R)$ if necessary, a replacement that does not change the conclusion of Lemma Lemma 1.3.2, it suffices to show `Rcut` under the assumption that G is contained in $B(o,R)$ and thus bounded. We make this assumption. For every $x \in G$ and $R_2 < R$, the domain of integration in `TRR` is contained in $B(o,2R)$. By replacing F with $F \cap B(o,2R)$ if necessary, and correspondingly restricting f to $B(o,2R)$, it suffices to show `Rcut` under the assumption that F is contained in $B(o,2R)$ and thus bounded. We make this assumption. Using the definition `defks` of K_s and the partition of unity `eq-psisum`, we observe that for fixed $R_1 < R_2$ we have

$$K(x,y)\mathbf{1}_{R_1<\rho(x,y)<R_2} = \sum_{s=s_1-2}^{s_2+2} K_s(x,y)\mathbf{1}_{R_1<\rho(x,y)<R_2},$$

where $s_1 = \lfloor \log_D 2R_1 \rfloor + 3$ and $s_2 = \lceil \log_D 4R_2 \rceil - 2$. To obtain this identity, we have used that on the set where $R_1 < \rho(x,y) < R_2$ the kernels K_s vanish unless s is inside the interval of summation. Similarly, we observe

$$\sum_{s=s_1}^{s_2} K_s(x,y)\mathbf{1}_{R_1<\rho(x,y)<R_2} = \sum_{s=s_1}^{s_2} K_s(x,y),$$

because on the support of K_s with $s_1 \le s \le s_2$ we have necessarily $R_1 < \rho(x,y) < R_2$. We thus express TRR as the sum of

$$T_{s_1,s_2} f(x) := \sum_{s_1 \le s \le s_2} \int K_s(x,y) f(y) e(Q(x)(y)) \, \mathrm{d}\mu(y)$$

and

$$\sum_{s=s_1-2, s_1-1, s_2+1, s_2+2} \int_{R_1 < \rho(x,y) < R_2} K_s(x,y) f(y) e(Q(x)(y)) \, \mathrm{d}\mu(y).$$

We apply the triangle inequality and estimate `Rcut` separately with $T_{R_1,R_2,R}$ replaced by the middle part and by the boundary part. To handle the case of the middle part, we employ Lemma 1.3.3. Here, we utilize the fact that if $\frac{1}{R} \le R_1 \le R_2 \le R$, then s_1 and s_2 are in an interval $[-S, S]$ for some sufficiently large S depending on R. To handle the case of the boundary part, we use the triangle inequality and the properties `supp-Ks`, `eq-Ks-size` of K_s and $|f| \le \mathbf{1}_F$ to obtain for arbitrary s the inequality

$$\left| \int_{R_1 < \rho(x,y) < R_2} K_s(x,y) f(y) e(Q(x)(y)) \, \mathrm{d}\mu(y) \right| \le \frac{2^{102a^3}}{\mu(B(x, D^s))} \int_{B(x,D^s)} \mathbf{1}_F(y) \, \mathrm{d}\mu(y) \le 2^{102a^3} M \mathbf{1}_F(x),$$

where $M\mathbf{1}_F$ is as defined in Theorem 1.2.6. Now the left-hand side of `Rcut`, with $T_{R_1,R_2,R}$ replaced by the boundary part, can be estimated using Holder's inequality and Theorem 1.2.6 by

$$2^{102a^3+2} \int \mathbf{1}_G(x) M \mathbf{1}_F(x) \, d\mu(x) \le \frac{2^{102a^3+4a+3} q}{q-1} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}}.$$

Summing the contributions from the middle and boundary parts completes the proof.

Lemma 1.3.3

Let F, G be bounded Borel sets in X. Let $f : X \to \mathbb{C}$ be a Borel function with $|f| \le 1_F$. Then for all $S \in \mathbb{Z}$ we have

$$\int \mathbf{1}_G(x) \sup_{-S \le s_1 \le s_2 \le S} |T_{s_1,s_2} f(x)| \, d\mu(x) \le \frac{2^{442a^3+2}}{(q-1)^6} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}},$$

where

$$T_{s_1,s_2} f(x) = \sum_{s_1 \le s \le s_2} \int_X K_s(x,y) f(y) e(Q(x)(y)) \, \mathrm{d}\mu(y).$$

Proof for Lemma 1.3.3

Proof of Lemma 1.3.3. We reduce Lemma 1.3.3 to Lemma 1.3.4. For each x, let $\sigma_1(x)$ be the minimal element $s' \in [-S, S]$ such that

$$\max_{s' \leq s_2 \leq S} |T_{s',s_2} f(x)| = \max_{-S \leq s_1 \leq s_2 \leq S} |T_{s_1,s_2} f(x)| =: T_{1,x}.$$

Similarly, let $\sigma_2(x)$ be the minimal element $s'' \in [-S, S]$ such that

$$|T_{\sigma_2(x),s''} f(x)| = T_{1,x}.$$

With these choices, and noting that with the definition of T_{2,σ_1,σ_2} from `middles1`

$$T_{\sigma_1(x),\sigma_2(x)} f(x) = T_{2,\sigma_1,\sigma_2} f(x),$$

we conclude that the left-hand side of `Scut` and `Sqlin` are equal. Therefore, Lemma 1.3.3 follows from Lemma 1.3.4.

Lemma 1.3.4

Let $\sigma_1, \sigma_2 \colon X \to \mathbb{Z}$ be measurable functions with finite range and $\sigma_1 \leq \sigma_2$. Then we have

$$\int \mathbf{1}_G(x) \, |T_{2,\sigma_1,\sigma_2} f(x)| \; d\mu(x) \leq \frac{2^{442a^3+2}}{(q-1)^6} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}},$$

with

$$T_{2,\sigma_1,\sigma_2} f(x) = \sum_{\sigma_1(x) \leq s \leq \sigma_2(x)} \int K_s(x,y) f(y) e(Q(x)(y)) \, \mathrm{d}\mu(y).$$

Proof for Lemma 1.3.4

Proof of Lemma 1.3.4. Fix σ_1, σ_2 and Q as in the lemma. Applying Theorem 1.2.1 recursively, we obtain a sequence of sets G_n with $G_0 = G$ and, for each $n \geq 0$, $G_{n+1} \subset G_n$, $\mu(G_n) \leq 2^{-n} \mu(G)$ and

$$\int \mathbf{1}_{G_n \setminus G_{n+1}}(x) \, |T_{2,\sigma_1,\sigma_2} f(x)| \; d\mu(x)$$
$$\leq \frac{2^{442a^3}}{(q-1)^5} \mu(G_n)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}}.$$

Adding the first n of these inequalities, we obtain by bounding a geometric series

$$\int \mathbf{1}_{G \setminus G_n}(x) \, |T_{2,\sigma_1,\sigma_2} f(x)| \; d\mu(x) \leq \frac{2^{442a^3+2}}{(q-1)^6} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}}.$$

As the integrand is non-negative and non-decreasing in n, we obtain by the monotone convergence theorem

$$\int \mathbf{1}_G(x) \, |T_{2,\sigma_1,\sigma_2} f(x)| \; d\mu(x) \leq \frac{2^{442a^3+2}}{(q-1)^6} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}}.$$

This completes the proof of Lemma 1.3.4 and thus Theorem 1.1.1.1.

1.4 Proof of Finitary Carleson

To prove Proposition Theorem 1.2.1, we already fixed in section 1.2 (p. 8) measurable functions σ_1, σ_2, Q and Borel sets F, G. We have also defined S to be the smallest integer such that the ranges of σ_1 and σ_2 are contained in $[-S, S]$ and F and G are contained in the ball $B(o, \frac{1}{4}D^S)$.

The proof of the next lemma is done in section 1.4 (p. 24), following the construction of dyadic cubes in the literature.

Lemma 1.4.1

There exists a grid structure $(\mathcal{D}, c, s)$.

The next lemma, which we prove in section 1.4 (p. 32), should be compared with the construction in the literature.

Lemma 1.4.2

For a given grid structure $(\mathcal{D}, c, s)$, there exists a tile structure $(\mathfrak{P}, \mathcal{I}, \Omega, \mathcal{Q}, \mathrm{c}, \mathrm{s})$.

Choose a grid structure $(\mathcal{D}, c, s)$ with Lemma 1.4.1 and a tile structure for this grid structure $(\mathfrak{P}, \mathcal{I}, \Omega, \mathcal{Q}, \mathrm{c}, \mathrm{s})$ with Lemma 1.4.2. Applying Theorem 1.2.2, we obtain a Borel set G' in X with $2\mu(G') \le \mu(G)$ such that for all Borel functions $f : X \to \mathbb{C}$ with $|f| \le \mathbf{1}_F$ we have `disclesssim`.

Lemma 1.4.3

We have for all $x \in G \setminus G'$

$$\sum_{\mathfrak{p}\in\mathfrak{P}} T_{\mathfrak{p}} f(x) = \sum_{s=\sigma_1(x)}^{\sigma_2(x)} \int K_s(x,y) f(y) e(Q(x)(y) - Q(x)(x))\, d\mu(y).$$

Proof for Lemma 1.4.3

Proof. Fix $x \in G \setminus G'$. Sorting the tiles $\mathfrak{p}$ on the left-hand side by the value $\mathrm{s}(\mathfrak{p}) \in [-S, S]$, it suffices to prove for every $-S \le s \le S$ that

$$\sum_{\mathfrak{p}\in\mathfrak{P}:\mathrm{s}(\mathfrak{p})=s} T_{\mathfrak{p}} f(x) = 0$$

if $s \notin [\sigma_1(x), \sigma_2(x)]$ and

$$\sum_{\mathfrak{p}\in\mathfrak{P}:\mathrm{s}(\mathfrak{p})=s} T_{\mathfrak{p}} f(x) = \int K_s(x,y) f(y) e(Q(x)(y) - Q(x)(x))\, d\mu(y).$$

if $s \in [\sigma_1(x), \sigma_2(x)]$. If $s \notin [\sigma_1(x), \sigma_2(x)]$, then by definition of $E(\mathfrak{p})$ we have $x \notin E(\mathfrak{p})$ for any $\mathfrak{p}$ with $\mathrm{s}(\mathfrak{p}) = s$ and thus $T_{\mathfrak{p}} f(x) = 0$. This proves the vanishing case. Now assume $s \in [\sigma_1(x), \sigma_2(x)]$. By `coverdyadic`, `subsetmaxcube`, `eq-vol-sp-cube`, the fact that $c(I_0) = o$ and $G \subset B(o, \frac{1}{4}D^S)$, there is at least one $I \in \mathcal{D}$ with

$s(I) = s$ and $x \in I$. By `dyadicproperty`, this I is unique. By `eq-dis-freq-cover`, there is precisely one $\mathfrak{p} \in \mathfrak{P}(I)$ such that $Q(x) \in \Omega(\mathfrak{p})$. Hence there is precisely one $\mathfrak{p} \in \mathfrak{P}$ with $\mathrm{s}(\mathfrak{p}) = s$ such that $x \in E(\mathfrak{p})$. For this $\mathfrak{p}$, the value $T_{\mathfrak{p}}(x)$ by its definition in `definetp` equals the right-hand side of the non-vanishing case. This proves the lemma.

We use this to prove Theorem 1.2.1.

Proof for Theorem 1.2.1

Proof of Theorem 1.2.1. We now estimate with Lemma 1.4.3 and Theorem 1.2.2

$$\int_{G\setminus G'} \left| \sum_{s=\sigma_1(x)}^{\sigma_2(x)} \int K_s(x,y) f(y) e(Q(x)(y)) \, \mathrm{d}\mu(y) \right| \mathrm{d}\mu(x)$$

$$= \int_{G\setminus G'} \left| \sum_{s=\sigma_1(x)}^{\sigma_2(x)} \int K_s(x,y) f(y) e(Q(x)(y) - Q(x)(x)) \mathrm{d}\mu(y) \right| \mathrm{d}\mu(x)$$

$$= \int_{G\setminus G'} \left| \sum_{\mathfrak{p}\in\mathfrak{P}} T_{\mathfrak{p}} f(x) \right| \mathrm{d}\mu(x) \le \frac{2^{442a^3}}{(q-1)^5} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}}.$$

This proves `eq-linearized` for the chosen set G' and arbitrary f and thus completes the proof of Proposition Theorem 1.2.1.

Proof of Grid Existence Lemma

We begin with the construction of the centers of the dyadic cubes.

Lemma 1.4.1.1

Let $-S \le k \le S$. Consider $Y \subset X$ such that for any $y \in Y$, we have

$$y \in B(o, 4D^S - D^k),$$

furthermore, for any $y' \in Y$ with $y \ne y'$, we have

$$B(y, D^k) \cap B(y', D^k) = \emptyset.$$

Then the cardinality of Y is bounded by

$$|Y| \le 2^{3a+200Sa^3}.$$

Proof for Lemma 1.4.1.1

Proof. Let k and Y be given. By applying the doubling property `doublingx` inductively, we have for each integer $j \geq 0$

$$\mu(B(y, 2^j D^k)) \leq 2^{aj} \mu(B(y, D^k)).$$

Since X is the union of the balls $B(y, 2^j D^k)$ and μ is not zero, at least one of the balls $B(y, 2^j D^k)$ has positive measure, thus $B(y, D^k)$ has positive measure. Applying the previous estimate for $j' = \ln_2(8D^{2S}) = 3 + 2S \cdot 100a^2$ by `defineD`, using $-S \leq k \leq S$, $y \in B(o, 4D^S)$, and the triangle inequality, we have

$$B(o, 4D^S) \subset B(y, 8D^S) \subset B(y, 2^{j'} D^k).$$

Using the disjointedness of the balls in `eq-disj-yballs`, `ybinb`, and the triangle inequality for ρ, we obtain

$$|Y|\mu(B(o, 4D^S)) \leq 2^{j'a} \sum_{y \in Y} \mu(B(y, D^k))$$

$$\leq 2^{j'a} \mu(\bigcup_{y \in Y} B(y, D^k)) \leq 2^{j'a} \mu(o, 4D^S).$$

As $\mu(o, 4D^S)$ is not zero, the lemma follows.

For each $-S \leq k \leq S$, let Y_k be a set of maximal cardinality in X such that $Y = Y_k$ satisfies the properties `ybinb` and `eq-disj-yballs` and such that $o \in Y_k$. By the upper bound of Lemma 1.4.1.1, such a set exists. For each $-S \leq k \leq S$, choose an enumeration of the points in the finite set Y_k and thus a total order $<$ on Y_k.

Lemma 1.4.1.2

For each $-S \leq k \leq S$, the ball $B(o, 4D^S - D^k)$ is contained in the union of the balls $B(y, 2D^k)$ with $y \in Y_k$.

Proof for Lemma 1.4.1.2

Proof. Let x be any point of $B(o, 4D^S - D^k)$. By maximality of $|Y_k|$, the ball $B(x, D^k)$ intersects one of the balls $B(y, D^k)$ with $y \in Y_k$. By the triangle inequality, $x \in B(y, 2D^k)$.

Define the set

$$\mathcal{C} := \{(y, k) : -S \leq k \leq S, y \in Y_k\},$$

We totally order the set $\mathcal{C}$ lexicographically by setting $(y, k) < (y', k')$ if $k < k'$ or both $k = k'$ and $y < y'$. In what follows, we define recursively in the sense of this order a function

$$(I_1, I_2, I_3) : \mathcal{C} \to \mathcal{P}(X) \times \mathcal{P}(X) \times \mathcal{P}(X).$$

Assume the sets $I_j(y', k')$ have already been defined for $j = 1, 2, 3$ if $k' < k$ and if $k = k'$ and $y' < y$. If $k = -S$, define for $j \in \{1, 2\}$ the set $I_j(y, k)$ to be $B(y, jD^{-S})$. If $-S < k$, define for $j \in \{1, 2\}$ and $y \in Y_k$ the set $I_j(y, k)$ to be

$$\bigcup\{I_3(y', k-1) : y' \in Y_{k-1} \cap B(y, jD^k)\}.$$

Define for $-S \leq k \leq S$ and $y \in Y_k$

$$I_3(y,k) := I_1(y,k) \cup \Big[I_2(y,k) \setminus \Big[X_k \cup \bigcup\{I_3(y',k) : y' \in Y_k, y' < y\}\Big)\Big]\Big]$$

with

$$X_k := \bigcup\{I_1(y', k) : y' \in Y_k\}.$$

Lemma 1.4.1.3

For each $-S \leq k \leq S$ and $1 \leq j \leq 3$ the following holds.If $j \neq 2$ and for some $x \in X$ and $y_1, y_2 \in Y_k$ we have

$$x \in I_j(y_1, k) \cap I_j(y_2, k),$$

then $y_1 = y_2$.If $j \neq 1$, then

$$B(o, 4D^S - 2D^k) \subset \bigcup_{y \in Y_k} I_j(y, k).$$

We have for each $y \in Y_k$,

$$B(y, \frac{1}{2}D^k) \subset I_3(y, k) \subset B(y, 4D^k).$$

Proof for Lemma 1.4.1.3

Proof. We prove these statements simultaneously by induction on the ordered set of pairs (y, k). Let $-S \leq k \leq S$. We first consider `disji` for $j = 1$. If $k = -S$, disjointedness of the sets $I_1(y, -S)$ follows by definition of I_1 and Y_k. If $k > -S$, assume x is in $I_1(y_m, k)$ for $m = 1, 2$. Then, for $m = 1, 2$, there is $z_m \in Y_{k-1} \cap B(y_m, D^k)$ with $x \in I_3(z_m, k-1)$. Using `disji` inductively for $j = 3$, we conclude $z_1 = z_2$. This implies that the balls $B(y_1, D^k)$ and $B(y_2, D^k)$ intersect. By construction of Y_k, this implies $y_1 = y_2$. This proves `disji` for $j = 1$. We next consider `disji` for $j = 3$. Assume x is in $I_3(y_m, k)$ for $m = 1, 2$ and $y_m \in Y_k$. If x is in X_k, then by definition `definei3`, $x \in I_1(y_m, k)$ for $m = 1, 2$. As we have already shown `disji` for $j = 1$, we conclude $y_1 = y_2$. This completes the proof in case $x \in X_k$, and we may assume x is not in X_k. By definition `definei3`, x is not in $I_3(z, k)$ for any z with $z < y_1$ or $z < y_2$. Hence, neither $y_1 < y_2$ nor $y_2 < y_1$, and by totality of the order of Y_k, we have $y_1 = y_2$. This completes the proof of `disji` for $j = 3$. We show `unioni` for $j = 2$. In case $k = -S$, this follows from Lemma 1.4.1.2. Assume $k > -S$. Let x be a point of $B(o, 4D^S - 2D^k)$. By induction, there is $y' \in Y_{k-1}$ such that $x \in I_3(y', k-1)$. Using the inductive statement `squeezedyadic`, we obtain $x \in B(y', 4D^{k-1})$. As $D > 4$, by applying the triangle inequality with the points o, x, and y', we obtain that $y' \in B(o, 4D^S - D^k)$. By Lemma 1.4.1.2, y' is in $B(y, 2D^k)$ for some $y \in Y_k$. It follows that $x \in I_2(y, k)$. This proves `unioni` for $j = 2$. We show `unioni` for $j = 3$. Let $x \in B(o, 4D^S - 2D^k)$. In case $x \in X_k$, then by definition of X_k we have $x \in I_1(y, k)$ for some $y \in Y_k$ and thus $x \in I_3(y, k)$. We may thus assume $x \notin X_k$. As we have already seen `unioni` for $j = 2$, there is $y \in Y_k$ such that $x \in I_2(y, k)$. We may assume this y is minimal with respect to the order in Y_k. Then $x \in I_3(y, k)$. This proves `unioni` for $j = 3$. Next, we show the first inclusion in `squeezedyadic`. Let $x \in B(y, \frac{1}{2}D^k)$. As $I_1(y, k) \subset I_3(y, k)$, it suffices to show $x \in I_1(y, k)$. If $k = -S$, this follows immediately from the assumption on x and the definition of I_1. Assume $k > -S$. By the inductive statement `unioni` and $D > 4$, there is a $y' \in Y_{k-1}$ such that $x \in I_3(y', k-1)$. By the inductive statement `squeezedyadic`, we conclude $x \in B(y', 4D^{k-1})$. By the triangle inequality with points x, y, y', and $D \geq 8$, we have $y' \in B(y, D^k)$. It follows by definition `defineij` that $I_3(y', k-1) \subset I_1(y, k)$, and thus $x \in I_3(y, k)$. This proves the first inclusion in `squeezedyadic`. We show the second inclusion in `squeezedyadic`. Let $x \in I_3(y, k)$. As $I_1(y, k) \subset I_2(y, k)$ directly from the definition `defineij`, it follows by definition `definei3` that $x \in I_2(y, k)$. By definition `defineij`, there is $y' \in Y_{k-1} \cap B(y, 2D^k)$ with $x \in I_3(y', k-1)$. By induction, $x \in B(y', 4D^{k-1})$. By the triangle inequality applied to the points x, y', y and $D > 4$, we conclude $x \in B(y, 4D^k)$. This shows the second inclusion in `squeezedyadic` and completes the proof of the lemma.

Lemma 1.4.1.4

Let $-S \leq l \leq k \leq S$ and $y \in Y_k$. We have

$$I_3(y, k) \subset \bigcup_{y' \in Y_l} I_3(y', l).$$

Proof for Lemma 1.4.1.4

Proof. Let $-S \leq l \leq k \leq S$ and $y \in Y_k$. If $l = k$, the inclusion `3coverdyadic` is true from the definition of set union. We may then assume inductively that $k > l$ and the statement of the lemma is true if k is replaced by $k - 1$. Let $x \in I_3(y, k)$. By definition `definei3`, $x \in I_j(y, k)$ for some $j \in \{1, 2\}$. By `defineij`, $x \in$

$I_3(w, k-1)$ for some $w \in Y_{k-1}$. We conclude `3coverdyadic` by induction.

Lemma 1.4.1.5

Let $-S \le l \le k \le S$ and $y \in Y_k$ and $y' \in Y_l$ with $I_3(y', l) \cap I_3(y, k) \neq \emptyset$. Then

$$I_3(y', l) \subset I_3(y, k).$$

Proof for Lemma 1.4.1.5

Proof. Let l, k, y, y' be as in the lemma. Pick $x \in I_3(y', l) \cap I_3(y, k)$. Assume first $l = k$. By `disji` of Lemma 1.4.1.3, we conclude $y' = y$, and thus `3dyadicproperty`. Now assume $l < k$. By induction, we may assume that the statement of the lemma is proven for $k-1$ in place of k. By Lemma 1.4.1.4, there is a $y'' \in Y_{k-1}$ such that $x \in I_3(y'', k-1)$. By induction, we have $I_3(y', l) \subset I_3(y'', k-1)$. It remains to prove

$$I_3(y'', k-1) \subset I_3(y, k).$$

We make a case distinction and assume first $x \in X_k$. By Definition `definei3`, we have $x \in I_1(y, k)$. By Definition `defineij`, there is a $v \in Y_{k-1} \cap B(y, D^k)$ with $x \in I_3(v, k-1)$. By `disji` of Lemma 1.4.1.3, we have $v = y''$. By Definition `defineij`, we then have $I_3(y'', k-1) \subset I_1(y, k)$. Then the claim follows by Definition `definei3` in the given case. Assume now the case $x \notin X_k$. By `definei3`, we have $x \in I_2(y, k)$. Moreover, for any $u < y$ in Y_k, we have $x \notin I_3(u, k)$. Let $u < y$. By transitivity of the order in Y_k, we conclude $x \notin I_2(u, k)$. By `defineij` and the disjointedness property of Lemma 1.4.1.3, we have $I_3(y'', k-1) \cap I_2(u, k) = \emptyset$. Similarly, $I_3(y'', k-1) \cap I_1(u, k) = \emptyset$. Hence $I_3(y'', k-1) \cap I_3(u, k) = \emptyset$. As $u < y$ was arbitrary, we conclude with `definei3` the claim in the given case. This completes the proof of the claim, and thus also `3dyadicproperty`.

For $-S \le k' \le k \le S$ and $y' \in Y_{k'}$, $y \in Y_k$ write $(y', k'|y, k)$ if $I_3(y', k') \subset I_3(y, k)$ and

$$\inf_{x \in X \setminus I_3(y,k)} \rho(y', x) < 6D^{k'}.$$

Lemma 1.4.1.6

Assume $-S \le k'' < k' < k \le S$ and $y'' \in Y_{k''}$, $y' \in Y_{k'}$, $y \in Y_k$. Assume there is $x \in X$ such that

$$x \in I_3(y'', k'') \cap I_3(y', k') \cap I_3(y, k).$$

If $(y'', k''|y, k)$, then also $(y'', k''|y', k')$ and $(y', k'|y, k)$.

Proof for Lemma 1.4.1.6

Proof. As $x \in I_3(y'', k'') \cap I_3(y', k')$ and $k'' < k'$, we have by Lemma 1.4.1.5 that $I_3(y'', k'') \subset I_3(y', k')$. Similarly, $I_3(y', k') \subset I_3(y, k)$. Pick $x' \in X \setminus I_3(y, k)$ such that $\rho(y'', x') < 6D^{k''}$, which exists as $(y'', k''|y, k)$. As $x' \in X \setminus I_3(y', k')$ as well, we conclude $(y'', k''|y', k')$. By the triangle inequality, we have

$$\rho(y', x') \leq \rho(y', x) + \rho(x, y'') + \rho(y'', x').$$

Using the choice of x and `squeezedyadic` as well as the choice of x', we estimate this by

$$< 4D^{k'} + 4D^{k''} + 6D^{k''} \leq 6D^{k'},$$

where we have used $D > 5$ and $k'' < k'$. We conclude $(y', k'|y, k)$.

Lemma 1.4.1.7

Let $K = 2^{4a+1}$. For each $-S + K \leq k \leq S$ and $y \in Y_k$ we have

$$\sum_{z \in Y_{k-K} : (z, k-K|y,k)} \mu(I_3(z, k-K)) \leq \frac{1}{2}\mu(I_3(y, k)).$$

Proof for Lemma 1.4.1.7

Proof. Let K be as in the lemma. Let $-S + K \leq k \leq S$ and $y \in Y_k$. Pick k' so that $k - K \leq k' \leq k$. For each $y'' \in Y_{k-K}$ with $(y'', k - K|y, k)$, by Lemma 1.4.1.4 and Lemma 1.4.1.5, there is a unique $y' \in Y_{k'}$ such that

$$I_3(y'', k - K) \subset I_3(y', k') \subset I_3(y, k).$$

Using Lemma 1.4.1.6, $(y', k'|y, k)$. We conclude using the disjointedness property of Lemma 1.4.1.3 that

$$\sum_{y'' : (y'', k-K|y,k)} \mu(I_3(y'', k - K)) \leq \sum_{y' : (y', k'|y,k)} \mu(I_3(y', k')).$$

Adding over $k - K < k' \leq k$, and using $\mu(I_3(y', k')) \leq 2^{4a}\mu(B(y', \frac{1}{4}D^{k'}))$ from `doublingx` and `squeezedyadic` gives

$$K \sum_{y'' : (y'', k-K|y,k)} \mu(I_3(y'', k-K)) \leq 2^{4a} \sum_{k-K < k' \leq k} \left[\sum_{y' : (y', k'|y,k)} \mu(B(y', \frac{1}{4}D^{k'})) \right].$$

Each ball $B(y', \frac{1}{4}D^{k'})$ occurring here is contained in $I_3(y', k')$ by `squeezedyadic` and in turn contained in $I_3(y, k)$. Assume for the moment all these balls are pairwise disjoint. Then by additivity of the measure,

$$K \sum_{y'':(y'',k-K|y,k)} \mu(I_3(y'', k-K)) \le 2^{4a}\mu(I_3(y,k)),$$

which by $K = 2^{4a+1}$ implies `new-small-boundary`. It thus remains to prove that the balls occurring in the previous sum are pairwise disjoint. Let (u, l) and (u', l') be two parameter pairs occurring in the sum and let $B(u, \frac{1}{4}D^l)$ and $B(u', \frac{1}{4}D^{l'})$ be the corresponding balls. If $l = l'$, then the balls are equal or disjoint by `squeezedyadic` and `disji` of Lemma 1.4.1.3. Assume then without loss of generality that $l' < l$. Towards a contradiction, assume that

$$B(u, \frac{1}{4}D^l) \cap B(u', \frac{1}{4}D^{l'}) \ne \emptyset.$$

As $(u', l'|y, k)$, there is a point x in $X \setminus I_3(y, k)$ with $\rho(x, u') < 6D^{l'}$. Using $D > 25$, we conclude from the triangle inequality and the assumed intersection that $x \in B(u, \frac{1}{2}D^l)$. However, $B(u, \frac{1}{2}D^l) \subset I_3(u, l)$, and $I_3(u, l) \subset I_3(y, k)$, a contradiction to $x \notin I_3(y, k)$. This proves the lemma.

Lemma 1.4.1.8

Let $K = 2^{4a+1}$ and let $n \ge 0$ be an integer. Then for each $-S + nK \le k \le S$ we have

$$\sum_{y' \in Y_{k-nK}:(y',k-nK|y,k)} \mu(I_3(y', k-nK)) \le 2^{-n}\mu(I_3(y,k)).$$

Proof for Lemma 1.4.1.8

Proof. We prove this by induction on n. If $n = 0$, both sides of `very-new-small` are equal to $\mu(I_3(y, k))$ by `disji`. If $n = 1$, this follows from Lemma 1.4.1.7. Assume $n > 1$ and `very-new-small` has been proven for $n - 1$. We write `very-new-small`

$$\sum_{y'' \in Y_{k-nK}:(y'',k-nK|y,k)} \mu(I_3(y'', k-nK))$$

$$= \sum_{y' \in Y_{k-K}:(y',k-K|y,k)} \left[\sum_{y'' \in Y_{k-nK}:(y'',k-nK|y',k-K)} \mu(I_3(y'', k-nK)) \right]$$

Applying the induction hypothesis, this is bounded by

$$= \sum_{y' \in Y_{k-K}:(y',k-K|y,k)} 2^{1-n}\mu(I_3(y', k-K))$$

Applying `new-small-boundary` gives `very-new-small`, and proves the lemma.

Lemma 1.4.1.9

For each $-S \le k \le S$ and $y \in Y_k$ and $0 < t < 1$ with $tD^k \ge D^{-S}$ we have

$$\mu(\{x \in I_3(y,k) \ : \ \rho(x, X \setminus I_3(y,k)) \le tD^k\}) \le 2t^\kappa \mu(I_3(y,k)).$$

Proof for Lemma 1.4.1.9

Proof. Let $x \in I_3(y,k)$ with $\rho(x, X \setminus I_3(y,k)) \le tD^k$. Let $K = 2^{4a+1}$ as in Lemma 1.4.1.8. Let n be the largest integer such that $D^{nK} \le \frac{1}{t}$, so that $tD^k \le D^{k-nK}$ and

$$D^{nK} > \frac{1}{tD^K}.$$

Let $k' = k - nK$, by the assumption $tD^k \ge D^{-S}$, we have $k' \ge -S$. By `3coverdyadic`, there exists $y' \in Y_{k'}$ with $x \in I_3(y', k')$. By the squeezing property `squeezedyadic` and the assumption on x, we have

$$\rho(y', X \setminus I_3(y,k)) \le \rho(x, y') + \rho(x, X \setminus I_3(y,k)) \le 4D^{k'} + tD^k.$$

By the assumption on n and the definition of k', this is

$$\le 4D^{k'} + D^{k-nK} < 6D^{k'}.$$

Together with `3dyadicproperty` thus $(y', k'|y,k)$. We have shown that

$$\{x \in I_3(y,k) \ : \ \rho(x, X \setminus I_3(y,k)) \le tD^k\}$$
$$\subset \bigcup_{y' \in Y_{k-nK} : (y', k-nK|y,k)} I_3(y', k-nK).$$

Using monotonicity and additivity of the measure and Lemma 1.4.1.8, we obtain

$$\mu(\{x \in I_3(y,k) \ : \ \rho(x, X \setminus I_3(y,k)) \le tD^k\}) \le 2^{-n} \mu(I_3(y,k)).$$

By `eq-n-size` and the definition `defineD` of D, this is bounded by

$$2t^{1/(100a^2K)} \mu(I_3(y,k)),$$

which completes the proof by the definition `definekappa` of κ.

Let $\tilde{\mathcal{D}}$ be the set of all $I_3(y,k)$ with $k \in [-S,S]$ and $y \in Y_k$. Define

$$s(I_3(y,k)) := k$$

$$c(I_3(y,k)) := y.$$

We define $\mathcal{D}$ to be the set of all $I \in \tilde{\mathcal{D}}$ such that $I \subset I_3(o,S)$.

Proof for Lemma 1.4.1

Proof of Lemma 1.4.1. We first show that $(\tilde{\mathcal{D}}, c, s)$ satisfies properties `coverdyadic`, `dyadicproperty`, `eq-vol-sp-cube` and `eq-small-boundary`. Property `eq-vol-sp-cube` follows from `squeezedyadic`, while `eq-small-boundary` follows from Lemma 1.4.1.9. Let $x \in B(o, D^S)$. We show properties `coverdyadic` and `dyadicproperty` for $(\tilde{\mathcal{D}}, c, s)$ and x. We first show `coverdyadic` for $(\tilde{\mathcal{D}}, c, s)$ by contradiction. Then there is an I violating the conclusion of `coverdyadic`. Pick such $I = I_3(y, l)$ such that l is minimal. By assumption, we have $-S \le k < l$; in particular $-S < l$. By definition, $I_3(y, l)$ is contained in $I_1(y, l) \cup I_2(y, l)$, which is contained in the union of $I_3(y', l-1)$ with $y' \in Y_{l-1}$. By minimality of l, each such $I_3(y', l-1)$ is contained in the union of all $I_3(z, k)$ with $z \in Y_k$. This proves `coverdyadic`. We now show `dyadicproperty` for $(\tilde{\mathcal{D}}, c, s)$. Assume to get a contradiction that there are non-disjoint $I, J \in \tilde{\mathcal{D}}$ with $s(I) \le s(J)$ and $I \not\subset J$. We may assume the existence of such I and J with minimal $s(J) - s(I)$. Let $k = s(I)$. Assume first $s(J) = k$. Let $I = I_3(y_1, k)$ and $J = I_3(y_2, k)$ with $y_1, y_2 \in Y_k$. If $y_1 = y_2$, then $I = J$, a contradiction to $I \not\subset J$. If $y_1 \ne y_2$, then $I \cap J = \emptyset$ by `disji`, a contradiction to the non-disjointedness of I, J. Assume now $s(J) > k$. Choose $y \in I \cap J$. By property `coverdyadic`, there is $K \in \tilde{\mathcal{D}}$ with $s(K) = s(J) - 1$ and $y \in K$. By construction of J, and pairwise disjointedness of all $I_3(w, s(J) - 1)$ that we have already seen, we have $K \subset J$. By minimality of $s(J)$, we have $I \subset K$. This proves $I \subset J$ and thus `dyadicproperty`. Now note that properties `dyadicproperty`, `eq-vol-sp-cube` and `eq-small-boundary` immediately carry over to $(\mathcal{D}, c, s)$ by restriction. `subsetmaxcube` is true for $(\mathcal{D}, c, s)$ by definition, and `coverdyadic` follows from `coverdyadic` and `dyadicproperty` for $(\tilde{\mathcal{D}}, c, s)$.

Proof of Tile Structure Lemma

Choose a grid structure $(\mathcal{D}, c, s)$ with Lemma 1.4.1. Let $I \in \mathcal{D}$. Suppose that

$$\mathcal{Z} \subset Q(X)$$

is such that for any $\vartheta, \theta \in \mathcal{Z}$ with $\vartheta \ne \theta$ we have

$$B_{I^\circ}(\vartheta, 0.3) \cap B_{I^\circ}(\theta, 0.3) \cap Q(X) = \emptyset.$$

Since $Q(X)$ is finite, there exists a set $\mathcal{Z}$ satisfying both conditions of maximal cardinality among all such sets. We pick for each $I \in \mathcal{D}$ such a set $\mathcal{Z}(I)$.
Lemma 1.4.2.1

For each $I \in \mathcal{D}$, we have

$$Q(X) \subset \bigcup_{z \in \mathcal{Z}(I)} B_{I^\circ}(z, 0.7).$$

Proof for Lemma 1.4.2.1

Proof. Let $\theta \in \bigcup_{\vartheta \in Q(X)} B_{I^\circ}(\vartheta, 1)$. By maximality of $\mathcal{Z}(I)$, there must be a point $z \in \mathcal{Z}(I)$ such that $B_{I^\circ}(z, 0.3) \cap B_{I^\circ}(\theta, 0.3) \neq \emptyset$. Else, $\mathcal{Z}(I) \cup \{\theta\}$ would be a set of larger cardinality than $\mathcal{Z}(I)$ satisfying `eq-tile-Z` and `eq-tile-disjoint-Z`. Fix such a z, and fix a point $z_1 \in B_{I^\circ}(z, 0.3) \cap B_{I^\circ}(\theta, 0.3)$. By the triangle inequality, we deduce that

$$d_{I^\circ}(z, \theta) \le d_{I^\circ}(z, z_1) + d_{I^\circ}(\theta, z_1) < 0.3 + 0.3 = 0.6,$$

and hence $\theta \in B_{I^\circ}(z, 0.7)$.

We define

$$\mathfrak{P} = \{(I, z) \ : \ I \in \mathcal{D}, z \in \mathcal{Z}(I)\},$$

$$\mathcal{I}((I, z)) = I \qquad \text{and} \qquad \mathcal{Q}((I, z)) = z.$$

We further set

$$\mathsf{s}(\mathfrak{p}) = s(\mathcal{I}(\mathfrak{p})), \qquad\qquad \mathsf{c}(\mathfrak{p}) = c(\mathcal{I}(\mathfrak{p})).$$

Then `tilecenter`, `tilescale` hold by definition. It remains to construct the map Ω, and verify properties `eq-dis-freq-cover`, `eq-freq-dyadic` and `eq-freq-comp-ball`. We first construct an auxiliary map Ω_1. For each $I \in \mathcal{D}$, we pick an enumeration of the finite set $\mathcal{Z}(I)$

$$\mathcal{Z}(I) = \{z_1, \dots, z_M\}.$$

We define $\Omega_1 : \mathfrak{P} \mapsto \mathcal{P}(\Theta)$ as follows. Set

$$\Omega_1((I, z_1)) = B_{I^\circ}(z_1, 0.7) \setminus \bigcup_{z \in \mathcal{Z}(I) \setminus \{z_1\}} B_{I^\circ}(z, 0.3)$$

and then define iteratively

$$\Omega_1((I, z_k)) = B_{I^\circ}(z_k, 0.7) \setminus \bigcup_{z \in \mathcal{Z}(I) \setminus \{z_k\}} B_{I^\circ}(z, 0.3) \setminus \bigcup_{i=1}^{k-1} \Omega_1((I, z_i)).$$

Lemma 1.4.2.2

For each $I \in \mathcal{D}$, and $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{P}(I)$, if

$$\Omega_1(\mathfrak{p}_1) \cap \Omega_1(\mathfrak{p}_2) \neq \emptyset$$

then $\mathfrak{p}_1 = \mathfrak{p}_2$.

Proof for Lemma 1.4.2.2

Proof. By the definition of the map $\mathcal{I}$, we have

$$\mathfrak{P}(I) = \{(I, z) \,:\, z \in \mathcal{Z}(I)\}.$$

By `eq-def-omega1`, the set $\Omega_1((I, z_k))$ is disjoint from each $\Omega_1((I, z_i))$ with $i < k$. Thus the sets $\Omega_1(\mathfrak{p})$, $\mathfrak{p} \in \mathfrak{P}(I)$ are pairwise disjoint.

Lemma 1.4.2.3

For each $I \in \mathcal{D}$, it holds that

$$\bigcup_{z \in \mathcal{Z}(I)} B_{I^\circ}(z, 0.7) \subset \bigcup_{\mathfrak{p} \in \mathfrak{P}(I)} \Omega_1(\mathfrak{p}).$$

For every $\mathfrak{p} \in \mathfrak{P}$, it holds that

$$B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 0.3) \subset \Omega_1(\mathfrak{p}) \subset B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 0.7).$$

Proof for Lemma 1.4.2.3

Proof. For `eq-omega1-incl` let $\mathfrak{p} = (I, z)$. The second inclusion in `eq-omega1-incl` then follows from `eq-def-omega1` and the equality $B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 0.7) = B_{I^\circ}(z, 0.7)$, which is true by definition. For the first inclusion in `eq-omega1-incl` let $\vartheta \in B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 0.3)$. Let k be such that $z = z_k$ in the enumeration chosen above. It follows immediately from `eq-def-omega1` and `eq-tile-disjoint-Z` that $\vartheta \notin \Omega_1((I, z_i))$ for all $i < k$. Thus, again from `eq-def-omega1`, we have $\vartheta \in \Omega_1((I, z_k))$. To show `eq-omega1-cover` let $\vartheta \in \bigcup_{z \in \mathcal{Z}(I)} B_{I^\circ}(z, 0.7)$. If there exists $z \in \mathcal{Z}(I)$ with $\vartheta \in B_{I^\circ}(z, 0.3)$, then

$$z \in \Omega_1((I, z)) \subset \bigcup_{\mathfrak{p} \in \mathfrak{P}(I)} \Omega_1(\mathfrak{p})$$

by the first inclusion in `eq-omega1-incl`. Now suppose that there exists no $z \in \mathcal{Z}(I)$ with $\vartheta \in B_{I^\circ}(z, 0.3)$. Let k be minimal such that $\vartheta \in B_{I^\circ}(z_k, 0.7)$. Since $\Omega_1((I, z_i)) \subset B_{I^\circ}(z_i, 0.7)$ for each i by `eq-def-omega1`, we have that $\vartheta \notin \Omega_1((I, z_i))$ for all $i < k$. Hence $\vartheta \in \Omega_1((I, z_k))$, again by `eq-def-omega1`.

Now we are ready to define the function Ω. We define for all $\mathfrak{p} \in \mathfrak{P}(I_0)$

$$\Omega(\mathfrak{p}) = \Omega_1(\mathfrak{p}).$$

For all other cubes $I \in \mathcal{D}$, $I \neq I_0$, there exists, by `dyadicproperty` and `subsetmaxcube`, $J \in \mathcal{D}$ with $I \subset J$ and $I \neq J$. On such I we define Ω by recursion. We can pick an inclusion minimal $J \in \mathcal{D}$ among the finitely many cubes such that $I \subset J$ and $I \neq J$. This J is unique. Suppose that J' is another inclusion minimal cube with $I \subset J'$ and $I \neq J'$. Without loss of generality, we have that $s(J) \le s(J')$. By `dyadicproperty`, it follows that $J \subset J'$. Since J' is minimal with respect to inclusion, it follows that $J = J'$. Then we define

$$\Omega(\mathfrak{p}) = \bigcup_{z \in \mathcal{Z}(J) \cap \Omega_1(\mathfrak{p})} \Omega((J, z)) \cup B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 0.2).$$

We now verify that $(\mathfrak{P}, \mathcal{I}, \Omega, \mathcal{Q}, \mathrm{c}, \mathrm{s})$ forms a tile structure.

Proof for Lemma 1.4.2

> Proof of Lemma 1.4.2. First, we prove `eq-freq-comp-ball`. If $I = I_0$, then `eq-freq-comp-ball` holds for all $\mathfrak{p} \in \mathfrak{P}(I)$ by `eq-max-omega` and `eq-omega1-incl`. Now suppose that I is not maximal in $\mathcal{D}$ with respect to set inclusion. Then we may assume by induction that for all $J \in \mathcal{D}$ with $I \subset J$ and all $\mathfrak{p}' \in \mathfrak{P}(J)$, `eq-freq-comp-ball` holds. Let J be the unique minimal cube in $\mathcal{D}$ with $I \subsetneq J$. Suppose that $\vartheta \in \Omega(\mathfrak{p})$. If $\vartheta \in B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 0.2)$, then since $B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 0.2) \subset B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 1)$, we conclude that $\vartheta \in B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 0.7)$. If not, by `eq-it-omega`, there exists $z \in \mathcal{Z}(J) \cap \Omega_1(\mathfrak{p})$ with $\vartheta \in \Omega(J, z)$. Using the triangle inequality, Lemma 1.2.1.2 and `eq-freq-comp-ball`, we obtain
>
> $$d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), z) \le d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \vartheta) + d_{\mathfrak{p}'}(z, \vartheta) \le 0.2 + 2^{-95a} \cdot 1 < 0.3.$$
>
> Thus $z \in \Omega_1(\mathfrak{p}')$ by `eq-omega1-incl`. By Lemma 1.4.2.2, it follows that $\mathfrak{p} = \mathfrak{p}'$. This shows the second inclusion in `eq-freq-comp-ball`. The first inclusion is immediate from `eq-it-omega`. Next, we show `eq-dis-freq-cover`. Let $I \in \mathcal{D}$. If $I = I_0$, then disjointedness of the sets $\Omega(\mathfrak{p})$ for $\mathfrak{p} \in \mathfrak{P}(I)$ follows from the definition `eq-max-omega` and Lemma 1.4.2.2. To obtain the inclusion in `eq-dis-freq-cover` one combines the inclusions `eq-tile-cover` and `eq-omega1-cover` of Lemma 1.4.2.3 with `eq-max-omega`. Now we turn to the case where there exists $J \in \mathcal{D}$ with $I \subset J$ and $I \neq J$. In this case we use induction: It suffices to show `eq-dis-freq-cover` under the assumption that it holds for all cubes $J \in \mathcal{D}$ with $I \subset J$. As shown before definition `eq-it-omega`, we may choose the unique inclusion minimal such J. To show disjointedness of the sets $\Omega(\mathfrak{p}), \mathfrak{p} \in \mathfrak{P}(I)$ we pick two tiles $\mathfrak{p}, \mathfrak{p}' \in \mathfrak{P}(I)$ and $\vartheta \in \Omega(\mathfrak{p}) \cap \Omega(\mathfrak{p}')$. Then we are by `eq-it-omega` in one of four cases. In the case where both come from elements $z, z' \in \mathcal{Z}(J)$, the induction hypothesis gives $z = z'$ and then Lemma 1.4.2.2 gives $\mathfrak{p} = \mathfrak{p}'$. In the mixed case where one side comes from $\Omega(J, z)$ and the other from $B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), 0.2)$, the same estimate as above gives $z \in \Omega_1(\mathfrak{p}')$, hence Lemma 1.4.2.2 gives $\mathfrak{p} = \mathfrak{p}'$. The symmetric mixed case is identical after swapping the roles. If both points lie in the small balls $B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 0.2) \cap B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), 0.2)$, then $\mathfrak{p} = \mathfrak{p}'$ because these small balls are pairwise disjoint by `eq-omega1-incl` and Lemma 1.4.2.2. To show the inclusion in `eq-dis-freq-cover`, let $\vartheta \in Q(X)$. By the induction

hypothesis, there exists $\mathfrak{p} \in \mathfrak{P}(J)$ such that $\vartheta \in \Omega(\mathfrak{p})$. By definition of the set $\mathfrak{P}$, we have $\mathfrak{p} = (J, z)$ for some $z \in \mathcal{Z}(J)$. Thus, by `eq-tile-cover`, there exists $z' \in \mathcal{Z}(I)$ with $z \in B_{I^\circ}(z', 0.7)$. Then by Lemma 1.4.2.3 there exists $\mathfrak{p}' \in \mathfrak{P}(I)$ with $z \in \mathcal{Z}(J) \cap \Omega_1(\mathfrak{p}')$. Consequently, by `eq-it-omega`, $\vartheta \in \Omega(\mathfrak{p}')$. This completes the proof of `eq-dis-freq-cover`. Finally, we show `eq-freq-dyadic`. Let $\mathfrak{p}, \mathfrak{q} \in \mathfrak{P}$ with $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{q})$ and $\Omega(\mathfrak{p}) \cap \Omega(\mathfrak{q}) \neq \emptyset$. If we have $\mathrm{s}(\mathfrak{p}) \geq \mathrm{s}(\mathfrak{q})$, then it follows from `dyadicproperty` that $I = J$, thus $\mathfrak{p}, \mathfrak{q} \in \mathfrak{P}(I)$. By `eq-dis-freq-cover` we have then either $\Omega(\mathfrak{p}) \cap \Omega(\mathfrak{q}) = \emptyset$ or $\Omega(\mathfrak{p}) = \Omega(\mathfrak{q})$. By the assumption in `eq-freq-dyadic` we have $\Omega(\mathfrak{p}) \cap \Omega(\mathfrak{q}) \neq \emptyset$, so we must have $\Omega(\mathfrak{p}) = \Omega(\mathfrak{q})$ and in particular $\Omega(\mathfrak{q}) \subset \Omega(\mathfrak{p})$. So it remains to show `eq-freq-dyadic` under the additional assumption that $\mathrm{s}(\mathfrak{q}) > \mathrm{s}(\mathfrak{p})$. In this case, we argue by induction on $\mathrm{s}(\mathfrak{q}) - \mathrm{s}(\mathfrak{p})$. By `coverdyadic`, there exists a cube $J \in \mathcal{D}$ with $s(J) = \mathrm{s}(\mathfrak{q}) - 1$ and $J \cap \mathcal{I}(\mathfrak{p}) \neq \emptyset$. We pick one such J. By `dyadicproperty`, we have $\mathcal{I}(\mathfrak{p}) \subset J \subset \mathcal{I}(\mathfrak{q})$. Thus, by `eq-tile-cover`, there exists $z' \in \mathcal{Z}(J)$ with $\mathcal{Q}(\mathfrak{q}) \in B_{J^\circ}(z', 0.7)$. Then by Lemma 1.4.2.3 there exists $\mathfrak{q}' \in \mathfrak{P}(J)$ with $\mathcal{Q}(\mathfrak{q}) \in \Omega_1(\mathfrak{q}')$. By `eq-it-omega`, it follows that $\Omega(\mathfrak{q}) \subset \Omega(\mathfrak{q}')$. Note that then $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{q}')$ and $\Omega(\mathfrak{p}) \cap \Omega(\mathfrak{q}') \neq \emptyset$ and $\mathrm{s}(\mathfrak{q}') - \mathrm{s}(\mathfrak{p}) = \mathrm{s}(\mathfrak{q}) - \mathrm{s}(\mathfrak{p}) - 1$. Thus, we have by the induction hypothesis that $\Omega(\mathfrak{q}') \subset \Omega(\mathfrak{p})$. This completes the proof.

1.5 Proof of discrete Carleson

Let a grid structure $(\mathcal{D}, c, s)$ and a tile structure $(\mathfrak{P}, \mathcal{I}, \Omega, \mathcal{Q})$ for this grid structure be given. In section 1.5 (p. 36), we decompose the set $\mathfrak{P}$ of tiles into subsets. Each subset will be controlled by one of three methods. The guiding principle of the decomposition is to be able to apply the forest estimate of Theorem 1.2.4 to the final subsets defined in `defc5`. This application is done in section 1.5 (p. 51). The miscellaneous subsets along the construction of the forests will either be thrown into exceptional sets, which are defined and controlled in section 1.5 (p. 40), or will be controlled by the antichain estimate of Theorem 1.2.3, which is done in section 1.5 (p. 56). section 1.5 (p. 46) contains some auxiliary lemmas needed for the proofs in Subsections section 1.5 (p. 51)-section 1.5 (p. 56).

Organisation of the tiles

In the following definitions, k, n, and j will be nonnegative integers. Define $\mathcal{C}(G, k)$ to be the set of $I \in \mathcal{D}$ such that there exists a $J \in \mathcal{D}$ with $I \subset J$ and

$$\mu(G \cap J) > 2^{-k-1} \mu(J),$$

but there does not exist a $J \in \mathcal{D}$ with $I \subset J$ and

$$\mu(G \cap J) > 2^{-k} \mu(J).$$

Let

$$\mathfrak{P}(k) = \{\mathfrak{p} \in \mathfrak{P} \ : \ \mathcal{I}(\mathfrak{p}) \in \mathcal{C}(G, k)\}$$

Define $\mathfrak{M}(k, n)$ to be the set of $\mathfrak{p} \in \mathfrak{P}(k)$ such that

$$\mu(E_1(\mathfrak{p})) > 2^{-n}\mu(\mathcal{I}(\mathfrak{p}))$$

and there does not exist $\mathfrak{p}' \in \mathfrak{P}(k)$ with $\mathfrak{p}' \neq \mathfrak{p}$ and $\mathfrak{p} \leq \mathfrak{p}'$ such that

$$\mu(E_1(\mathfrak{p}')) > 2^{-n}\mu(\mathcal{I}(\mathfrak{p}')).$$

Define for a collection $\mathfrak{P}' \subset \mathfrak{P}(k)$

$$\text{dens}'_k(\mathfrak{P}') := \sup_{\mathfrak{p}' \in \mathfrak{P}'} \sup_{\lambda \geq 2} \lambda^{-a} \sup_{\mathfrak{p} \in \mathfrak{P}(k) : \lambda\mathfrak{p}' \lesssim \lambda\mathfrak{p}} \frac{\mu(E_2(\lambda, \mathfrak{p}))}{\mu(\mathcal{I}(\mathfrak{p}))}.$$

Sorting by density, we define

$$\mathfrak{C}(k, n) := \{\mathfrak{p} \in \mathfrak{P}(k) \ : \ 2^{4a}2^{-n} < \text{dens}'_k(\{\mathfrak{p}\}) \leq 2^{4a}2^{-n+1}\}.$$

Following Fefferman, we define for $\mathfrak{p} \in \mathfrak{C}(k, n)$

$$\mathfrak{B}(\mathfrak{p}) := \{\mathfrak{m} \in \mathfrak{M}(k, n) \ : \ 100\mathfrak{p} \lesssim \mathfrak{m}\}$$

and

$$\mathfrak{C}_1(k, n, j) := \{\mathfrak{p} \in \mathfrak{C}(k, n) \ : \ 2^j \leq |\mathfrak{B}(\mathfrak{p})| < 2^{j+1}\},$$

and

$$\mathfrak{L}_0(k, n) := \{\mathfrak{p} \in \mathfrak{C}(k, n) \ : \ |\mathfrak{B}(\mathfrak{p})| < 1\}.$$

Together with the following removal of minimal layers, the splitting into $\mathfrak{C}_1(k, n, j)$ will lead to a separation of trees. Define recursively for $0 \leq l \leq Z(n + 1)$

$$\mathfrak{L}_1(k, n, j, l)$$

to be the set of minimal elements with respect to $\leq$ in

$$\mathfrak{C}_1(k, n, j) \setminus \bigcup_{0 \leq l' < l} \mathfrak{L}_1(k, n, j, l').$$

Define

$$\mathfrak{C}_2(k, n, j) := \mathfrak{C}_1(k, n, j) \setminus \bigcup_{0 \leq l' \leq Z(n+1)} \mathfrak{L}_1(k, n, j, l').$$

The remaining tile organization will be relative to prospective tree tops, which we define now. Define

$$\mathfrak{U}_1(k, n, j)$$

to be the set of all $\mathfrak{u} \in \mathfrak{C}_1(k, n, j)$ such that for all $\mathfrak{p} \in \mathfrak{C}_1(k, n, j)$ with $\mathcal{I}(\mathfrak{u})$ strictly contained in $\mathcal{I}(\mathfrak{p})$ we have $B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), 100) \cap B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 100) = \emptyset$. We first remove the pairs that are outside the immediate reach of any of the prospective tree tops. Define

$$\mathfrak{L}_2(k,n,j)$$

to be the set of all $\mathfrak{p} \in \mathfrak{C}_2(k,n,j)$ such that there does not exist $\mathfrak{u} \in \mathfrak{U}_1(k,n,j)$ with $\mathcal{I}(\mathfrak{p}) \neq \mathcal{I}(\mathfrak{u})$ and $2\mathfrak{p} \lesssim \mathfrak{u}$. Define

$$\mathfrak{C}_3(k,n,j) := \mathfrak{C}_2(k,n,j) \setminus \mathfrak{L}_2(k,n,j).$$

We next remove the maximal layers. Define recursively for $0 \leq l \leq Z(n+1)$

$$\mathfrak{L}_3(k,n,j,l)$$

to be the set of all maximal elements with respect to $\leq$ in

$$\mathfrak{C}_3(k,n,j) \setminus \bigcup_{0 \leq l' < l} \mathfrak{L}_3(k,n,j,l').$$

Define

$$\mathfrak{C}_4(k,n,j) := \mathfrak{C}_3(k,n,j) \setminus \bigcup_{0 \leq l \leq Z(n+1)} \mathfrak{L}_3(k,n,j,l).$$

Finally, we remove the boundary pairs relative to the prospective tree tops. Define

$$\mathcal{L}(\mathfrak{u})$$

to be the set of all $I \in \mathcal{D}$ with $I \subset \mathcal{I}(\mathfrak{u})$ and $s(I) = \mathrm{s}(\mathfrak{u}) - Z(n+1) - 1$ and $B(c(I), 8D^{s(I)}) \not\subset \mathcal{I}(\mathfrak{u})$. Define

$$\mathfrak{L}_4(k,n,j)$$

to be the set of all $\mathfrak{p} \in \mathfrak{C}_4(k,n,j)$ such that there exists $\mathfrak{u} \in \mathfrak{U}_1(k,n,j)$ with $\mathcal{I}(\mathfrak{p}) \subset \bigcup \mathcal{L}(\mathfrak{u})$, and define

$$\mathfrak{C}_5(k,n,j) := \mathfrak{C}_4(k,n,j) \setminus \mathfrak{L}_4(k,n,j).$$

We define three exceptional sets. The first exceptional set G_1 takes into account the ratio of the measures of F and G. Define $\mathfrak{P}_{F,G}$ to be the set of all $\mathfrak{p} \in \mathfrak{P}$ with

$$\mathrm{dens}_2(\{\mathfrak{p}\}) > 2^{2a+5} \frac{\mu(F)}{\mu(G)}.$$

Define

$$G_1 := \bigcup_{\mathfrak{p} \in \mathfrak{P}_{F,G}} \mathcal{I}(\mathfrak{p}).$$

For an integer $\lambda \geq 0$, define $A(\lambda, k, n)$ to be the set of all $x \in X$ such that

$$\sum_{\mathfrak{p} \in \mathfrak{M}(k,n)} \mathbf{1}_{\mathcal{I}(\mathfrak{p})}(x) > \lambda 2^{n+1}$$

and define

$$G_2 := \bigcup_{k\geq 0} \bigcup_{k\leq n} A(2n+6,k,n).$$

Define

$$G_3 := \bigcup_{k\geq 0} \bigcup_{n\geq k} \bigcup_{0\leq j\leq 2n+3} \bigcup_{\mathfrak{p}\in\mathfrak{L}_4(k,n,j)} \mathcal{I}(\mathfrak{p}).$$

Define $G' = G_1 \cup G_2 \cup G_3$. The following bound of the measure of G' will be proven in section 1.5 (p. 40).

Lemma 1.5.1.1

We have

$$\mu(G') \leq 2^{-1}\mu(G).$$

In section 1.5 (p. 51), we identify each set $\mathfrak{C}_5(k,n,j)$ outside G' as a forest and use Proposition Theorem 1.2.4 to prove the following lemma.

Lemma 1.5.1.2

Let

$$\mathfrak{P}_1 = \bigcup_{k\geq 0} \bigcup_{n\geq k} \bigcup_{0\leq j\leq 2n+3} \mathfrak{C}_5(k,n,j)$$

For all $f : X \to \mathbb{C}$ with $|f| \leq \mathbf{1}_F$ we have

$$\int_{G\setminus G'} \left| \sum_{\mathfrak{p}\in\mathfrak{P}_1} T_{\mathfrak{p}} f \right| \,\mathrm{d}\mu \leq \frac{2^{441a^3}}{(q-1)^4} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}}.$$

In section 1.5 (p. 56), we decompose the complement of the set of tiles in Lemma Lemma 1.5.1.2 and apply the antichain estimate of Theorem 1.2.3 to prove the following lemma.

Lemma 1.5.1.3

Let

$$\mathfrak{P}_2 = \mathfrak{P} \setminus \mathfrak{P}_1.$$

For all $f : X \to \mathbb{C}$ with $|f| \leq \mathbf{1}_F$ we have

$$\int_{G\setminus G'} \left| \sum_{\mathfrak{p}\in\mathfrak{P}_2} T_{\mathfrak{p}} f \right| \,\mathrm{d}\mu \leq \frac{2^{120a^3}}{(q-1)^5} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}}.$$

Proof for Theorem 1.2.2

Proof of Theorem 1.2.2. Theorem 1.2.2 follows by applying the triangle inequality to `disclesssim` according to the splitting in Lemma 1.5.1.2 and Lemma 1.5.1.3 and using both lemmas as well as the bound on the set G' given by Lemma 1.5.1.1.

Proof of the Exceptional Sets Lemma

We prove separate bounds for G_1, G_2, and G_3 in Lemmas Lemma 1.5.2.1, Lemma 1.5.2.6, and Lemma 1.5.2.10. Adding up these bounds proves Lemma 1.5.1.1. The bound for G_1 follows from the Vitali covering lemma, Theorem 1.2.6.
Lemma 1.5.2.1

We have
$$\mu(G_1) \le 2^{-5}\mu(G).$$

Proof for Lemma 1.5.2.1

Proof. Let
$$K = 2^{2a+5}\frac{\mu(F)}{\mu(G)}.$$

For each $\mathfrak{p} \in \mathfrak{P}_{F,G}$ pick a $r(\mathfrak{p}) > 4D^{\mathsf{s}(\mathfrak{p})}$ with
$$\mu(F \cap B(\mathsf{c}(\mathfrak{p}), r(\mathfrak{p}))) \ge K\mu(B(\mathsf{c}(\mathfrak{p}), r(\mathfrak{p}))).$$

This ball exists by definition of $\mathfrak{P}_{F,G}$ and `dens_2`. By applying Theorem 1.2.6 to the collection of balls
$$\mathcal{B} = \{B(\mathsf{c}(\mathfrak{p}), r(\mathfrak{p})) \ : \ \mathfrak{p} \in \mathfrak{P}_{F,G}\}$$

and the function $u = \mathbf{1}_F$, we obtain
$$\mu(\bigcup \mathcal{B}) \le 2^{2a}K^{-1}\mu(F).$$

We conclude with `eq-vol-sp-cube` and $r(\mathfrak{p}) > 4D^{\mathsf{s}(\mathfrak{p})}$
$$\mu(G_1) = \mu(\bigcup_{\mathfrak{p}\in\mathfrak{P}_{F,G}} \mathcal{I}(\mathfrak{p})) \le \mu(\bigcup \mathcal{B}) \le 2^{2a}K^{-1}\mu(F) = 2^{-5}\mu(G).$$

We turn to the bound of G_2, which relies on the Dyadic Covering Lemma 1.5.2.2 and the John-Nirenberg Lemma 1.5.2.5 below.
Lemma 1.5.2.2

For each $k \ge 0$, the union of all dyadic cubes in $\mathcal{C}(G, k)$ has measure at most $2^{k+1}\mu(G)$.

Proof for Lemma 1.5.2.2

Proof. The union of dyadic cubes in $\mathcal{C}(G, k)$ is contained in the union of elements of the set $\mathcal{M}(k)$ of all dyadic cubes J with $\mu(G \cap J) > 2^{-k-1}\mu(J)$. The union of elements in the set $\mathcal{M}(k)$ is contained in the union of elements in the set $\mathcal{M}^*(k)$ of maximal elements in $\mathcal{M}(k)$ with respect to set inclusion. Hence

$$\mu(\bigcup \mathcal{C}(G,k)) \le \mu(\bigcup \mathcal{M}^*(k)) \le \sum_{J\in\mathcal{M}^*(k)} \mu(J).$$

Using the definition of $\mathcal{M}(k)$ and then the pairwise disjointedness of elements in $\mathcal{M}^*(k)$, we estimate this by

$$\le 2^{k+1} \sum_{J\in\mathcal{M}^*(k)} \mu(J\cap G) \le 2^{k+1}\mu(G).$$

This proves the lemma.

Lemma 1.5.2.3

If $\mathfrak{p}, \mathfrak{p}' \in \mathfrak{M}(k,n)$ and

$$E_1(\mathfrak{p}) \cap E_1(\mathfrak{p}') \neq \emptyset$$

then $\mathfrak{p} = \mathfrak{p}'$.

Proof for Lemma 1.5.2.3

Proof. Let $\mathfrak{p}, \mathfrak{p}'$ be as in the lemma. By definition of E_1, we have $E_1(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{p})$ and analogously for $\mathfrak{p}'$, we conclude from `eintersect` that $\mathcal{I}(\mathfrak{p}) \cap \mathcal{I}(\mathfrak{p}') \neq \emptyset$. Let without loss of generality $\mathcal{I}(\mathfrak{p})$ be maximal in $\{\mathcal{I}(\mathfrak{p}), \mathcal{I}(\mathfrak{p}')\}$, then $\mathcal{I}(\mathfrak{p}') \subset \mathcal{I}(\mathfrak{p})$. By `eintersect`, we conclude by definition of E_1 that $\Omega(\mathfrak{p})\cap\Omega(\mathfrak{p}') \neq \emptyset$. By `eq-freq-dyadic` we conclude $\Omega(\mathfrak{p}) \subset \Omega(\mathfrak{p}')$. It follows that $\mathfrak{p}' \le \mathfrak{p}$. By maximality `mnkmax` of $\mathfrak{p}'$, we have $\mathfrak{p}' = \mathfrak{p}$. This proves the lemma.

Lemma 1.5.2.4

For each $x \in A(\lambda, k, n)$, there is a dyadic cube I that contains x and is a subset of $A(\lambda, k, n)$.

Proof for Lemma 1.5.2.4

Proof. Fix k, n, λ, x as in the lemma such that $x \in A(\lambda, k, n)$. Let $\mathcal{M}$ be the set of dyadic cubes $\mathcal{I}(\mathfrak{p})$ with $\mathfrak{p}$ in $\mathfrak{M}(k,n)$ and $x \in \mathcal{I}(\mathfrak{p})$. By definition of $A(\lambda, k, n)$, the cardinality of $\mathcal{M}$ is at least $1 + \lambda 2^{n+1}$. Let I be a cube of smallest scale in $\mathcal{M}$. Then I is contained in all cubes of $\mathcal{M}$. It follows that $I \subset A(\lambda, k, n)$.

Lemma 1.5.2.5

For all integers $k, n, \lambda \ge 0$, we have

$$\mu(A(\lambda, k, n)) \le 2^{k+1-\lambda}\mu(G).$$

Proof for Lemma 1.5.2.5

Proof. Fix k, n as in the lemma and suppress notation to write $A(\lambda)$ for $A(\lambda, k, n)$. We prove the lemma by induction on λ. For $\lambda = 0$, we use that $A(\lambda)$ by definition of $\mathfrak{M}(k, n)$ is contained in the union of elements in $\mathcal{C}(G, k)$. Lemma 1.5.2.2 then completes the base of the induction. Now assume that the statement of Lemma 1.5.2.5 is proven for some integer $\lambda \geq 0$. The set $A(\lambda + 1)$ is contained in the set $A(\lambda)$. Let $\mathcal{M}$ be the set of dyadic cubes which are a subset of $A(\lambda)$. By Lemma 1.5.2.4, the union of $\mathcal{M}$ is $A(\lambda)$. Let $\mathcal{M}^*$ be the set of maximal dyadic cubes in $\mathcal{M}$. Let $x \in A(\lambda + 1)$ and $L \in \mathcal{M}^*$ such that $x \in L$. Then by the dyadic property `dyadicproperty`

$$\sum_{\mathfrak{p} \in \mathfrak{M}(k,n)} \mathbf{1}_{\mathcal{I}(\mathfrak{p})}(x) = \sum_{\mathfrak{p} \in \mathfrak{M}(k,n): \mathcal{I}(\mathfrak{p}) \subset L} \mathbf{1}_{\mathcal{I}(\mathfrak{p})}(x) + \sum_{\mathfrak{p} \in \mathfrak{M}(k,n): L \subsetneq \mathcal{I}(\mathfrak{p})} \mathbf{1}_{\mathcal{I}(\mathfrak{p})}(x).$$

We now show

$$\sum_{\mathfrak{p} \in \mathfrak{M}(k,n): \mathcal{I}(\mathfrak{p}) \subset L} \mathbf{1}_{\mathcal{I}(\mathfrak{p})}(x) \geq 2^{n+1}.$$

The left-hand side of the previous display is strictly greater than $(\lambda + 1)2^{n+1}$. If L is the top cube the second sum is zero and the claim follows immediately. Otherwise consider the inclusion-minimal cube $\hat{L}$ with $L \subsetneq \hat{L}$; all tiles in the second sum satisfy $\hat{L} \subset \mathfrak{p}$, so the second sum is constant for all $x \in \hat{L}$. By maximality of L, the second sum is at most $\lambda 2^{n+1}$ somewhere on $\hat{L}$, thus on all of $\hat{L}$ and consequently also at x. Rearranging the inequality yields the claim. By Lemma 1.5.2.3, we have

$$\sum_{\mathfrak{p} \in \mathfrak{M}(k,n): \mathcal{I}(\mathfrak{p}) \subset L} \mu(E_1(\mathfrak{p})) \leq \mu(L).$$

Multiplying by 2^n and applying `ebardense`, we obtain

$$\sum_{\mathfrak{p} \in \mathfrak{M}(k,n): \mathcal{I}(\mathfrak{p}) \subset L} \mu(\mathcal{I}(\mathfrak{p})) \leq 2^n \mu(L).$$

We then have with the previous two estimates

$$2^{n+1} \mu(A(\lambda + 1) \cap L) = \int_{A(\lambda+1) \cap L} 2^{n+1} d\mu$$

$$\leq \int \sum_{\mathfrak{p} \in \mathfrak{M}(k,n): \mathcal{I}(\mathfrak{p}) \subset L} \mathbf{1}_{\mathcal{I}(\mathfrak{p})} d\mu \leq 2^n \mu(L).$$

Hence

$$2\mu(A(\lambda + 1)) = 2 \sum_{L \in \mathcal{M}^*} \mu(A(\lambda + 1) \cap L) \leq \sum_{L \in \mathcal{M}^*} \mu(L) = \mu(A(\lambda)).$$

Using the induction hypothesis, this proves `alambdameasure` for $\lambda + 1$ and completes the proof of the lemma.

Lemma 1.5.2.6

We have

$$\mu(G_2) \le 2^{-2}\mu(G).$$

Proof for Lemma 1.5.2.6

Proof. We use Lemma 1.5.2.5 and sum twice a geometric series to obtain

$$\sum_{0\le k}\sum_{k\le n}\mu(A(2n+6,k,n)) \le \sum_{0\le k}\sum_{k\le n}2^{k-5-2n}\mu(G)$$

$$\le \sum_{0\le k}2^{-k-3}\mu(G) \le 2^{-2}\mu(G).$$

This proves the lemma.

We turn to the set G_3.

Lemma 1.5.2.7

We have

$$\sum_{\mathfrak{m}\in\mathfrak{M}(k,n)}\mu(\mathcal{I}(\mathfrak{m})) \le 2^{n+k+3}\mu(G).$$

Proof for Lemma 1.5.2.7

Proof. We write the left-hand side of `eq-musum`

$$\int \sum_{\mathfrak{m}\in\mathfrak{M}(k,n)}\mathbf{1}_{\mathcal{I}(\mathfrak{m})}(x)\,d\mu(x) \le 2^{n+1}\sum_{\lambda=0}^{|\mathfrak{M}|}\mu(A(\lambda,k,n)).$$

Using Lemma 1.5.2.5 and then summing a geometric series, we estimate this by

$$\le 2^{n+1}\sum_{\lambda=0}^{|\mathfrak{M}|}2^{k+1-\lambda}\mu(G) \le 2^{n+1}2^{k+2}\mu(G).$$

This proves the lemma.

Lemma 1.5.2.8

Let $k,n,j\ge 0$. We have for every $x\in X$

$$\sum_{\mathfrak{u}\in\mathfrak{U}_1(k,n,j)}\mathbf{1}_{\mathcal{I}(\mathfrak{u})}(x) \le 2^{-j}2^{9a}\sum_{\mathfrak{m}\in\mathfrak{M}(k,n)}\mathbf{1}_{\mathcal{I}(\mathfrak{m})}(x).$$

Proof for Lemma 1.5.2.8

Proof. Let $x \in X$. For each $\mathfrak{u} \in \mathfrak{U}_1(k,n,j)$ with $x \in \mathcal{I}(\mathfrak{u})$, as $\mathfrak{u} \in \mathfrak{C}_1(k,n,j)$, there are at least 2^j elements $\mathfrak{m} \in \mathfrak{M}(k,n)$ with $100\mathfrak{u} \lesssim \mathfrak{m}$ and in particular $x \in \mathcal{I}(\mathfrak{m})$. Hence

$$\mathbf{1}_{\mathcal{I}(\mathfrak{u})}(x) \le 2^{-j} \sum_{\mathfrak{m} \in \mathfrak{M}(k,n): 100\mathfrak{u} \lesssim \mathfrak{m}} \mathbf{1}_{\mathcal{I}(\mathfrak{m})}(x).$$

Conversely, for each $\mathfrak{m} \in \mathfrak{M}(k,n)$ with $x \in \mathcal{I}(\mathfrak{m})$, let $\mathfrak{U}(\mathfrak{m})$ be the set of $\mathfrak{u} \in \mathfrak{U}_1(k,n,j)$ with $x \in \mathcal{I}(\mathfrak{u})$ and $100\mathfrak{u} \lesssim \mathfrak{m}$. Summing the previous bound over $\mathfrak{u}$ and counting the pairs $(\mathfrak{u}, \mathfrak{m})$ with $100\mathfrak{u} \lesssim \mathfrak{m}$ differently gives

$$\sum_{\mathfrak{u} \in \mathfrak{U}_1(k,n,j)} \mathbf{1}_{\mathcal{I}(\mathfrak{u})}(x) \le 2^{-j} \sum_{\mathfrak{m} \in \mathfrak{M}(k,n)} \sum_{\mathfrak{u} \in \mathfrak{U}(\mathfrak{m})} \mathbf{1}_{\mathcal{I}(\mathfrak{m})}(x).$$

We estimate the number of elements in $\mathfrak{U}(\mathfrak{m})$. Let $\mathfrak{u} \in \mathfrak{U}(\mathfrak{m})$. Then by definition of $\mathfrak{U}(\mathfrak{m})$

$$d_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), \mathcal{Q}(\mathfrak{m})) \le 100.$$

If $\mathfrak{u}'$ is a further element in $\mathfrak{U}(\mathfrak{m})$ with $\mathfrak{u} \neq \mathfrak{u}'$, then

$$\mathcal{Q}(\mathfrak{m}) \in B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), 100) \cap B_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), 100).$$

By the last display and definition of $\mathfrak{U}_1(k,n,j)$, none of $\mathcal{I}(\mathfrak{u}), \mathcal{I}(\mathfrak{u}')$ is strictly contained in the other. As both contain x, we have $\mathcal{I}(\mathfrak{u}) = \mathcal{I}(\mathfrak{u}')$. We then have $d_{\mathfrak{u}} = d_{\mathfrak{u}'}$. By `eq-freq-comp-ball`, the balls $B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), 0.2)$ and $B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}'), 0.2)$ are contained respectively in $\Omega(\mathfrak{u})$ and $\Omega(\mathfrak{u}')$ and thus are disjoint by `eq-dis-freq-cover`. By `dby2` and the triangle inequality, both balls are contained in $B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{m}), 100.2)$. By `thirddb` applied nine times, there is a collection of at most 2^{9a} balls of radius 0.2 with respect to the metric $d_{\mathfrak{u}}$ which cover the ball $B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{m}), 100.2)$. Let B' be a ball in this cover. As the center of B' can be in at most one of the disjoint balls $B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), 0.2)$ and $B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}'), 0.2)$, the ball B' can contain at most one of the points $\mathcal{Q}(\mathfrak{u})$, $\mathcal{Q}(\mathfrak{u}')$. Hence the image of $\mathfrak{U}(\mathfrak{m})$ under $\mathcal{Q}$ has at most 2^{9a} elements; since $\mathcal{Q}$ is injective on $\mathfrak{U}(\mathfrak{m})$, the same is true of $\mathfrak{U}(\mathfrak{m})$. Inserting this into `usumbymsum` proves the lemma.

Lemma 1.5.2.9

Let $\mathcal{L}(\mathfrak{u})$ be as defined in `eq-L-def`. We have for each $\mathfrak{u} \in \mathfrak{U}_1(k,n,l)$,

$$\mu(\bigcup_{I \in \mathcal{L}(\mathfrak{u})} I) \le D^{1-\kappa Z(n+1)} \mu(\mathcal{I}(\mathfrak{u})).$$

Proof for Lemma 1.5.2.9

Proof. Let $\mathfrak{u} \in \mathfrak{U}_1(k,n,l)$. Let $I \in \mathcal{L}(\mathfrak{u})$. Then we have $s(I) = \mathrm{s}(\mathfrak{u}) - Z(n+1) - 1$ and $I \subset \mathcal{I}(\mathfrak{u})$ and $B(c(I), 8D^{s(I)}) \not\subset \mathcal{I}(\mathfrak{u})$. By `eq-vol-sp-cube`, the set I is contained in $B(c(I), 4D^{s(I)})$. By the triangle inequality, the set I is contained in

$$X(\mathfrak{u}) := \{x \in \mathcal{I}(\mathfrak{u}) \,:\, \rho(x, X \setminus \mathcal{I}(\mathfrak{u})) \le 12D^{\mathrm{s}(\mathfrak{u})-Z(n+1)-1}\}.$$

By the small boundary property `eq-small-boundary`, noting that $12D^{\mathrm{s}(\mathfrak{u})-Z(n+1)-1} = 12D^{s(I)} > D^{-S}$, we have

$$\mu(X(\mathfrak{u})) \le 2 \cdot (12D^{-Z(n+1)-1})^{\kappa} \mu(\mathcal{I}(\mathfrak{u})).$$

Using $\kappa < 1$ and $D \ge 12$, this proves the lemma.

Lemma 1.5.2.10

We have

$$\mu(G_3) \le 2^{-4}\mu(G).$$

Proof for Lemma 1.5.2.10

Proof. As each $\mathfrak{p} \in \mathfrak{L}_4(k,n,j)$ is contained in $\cup\mathcal{L}(\mathfrak{u})$ for some $\mathfrak{u} \in \mathfrak{U}_1(k,n,l)$, we have

$$\mu(\bigcup_{\mathfrak{p}\in\mathfrak{L}_4(k,n,j)} \mathcal{I}(\mathfrak{p})) \le \sum_{\mathfrak{u}\in\mathfrak{U}_1(k,n,j)} \mu(\bigcup_{I\in\mathcal{L}(\mathfrak{u})} I).$$

Using Lemma 1.5.2.9 and then Lemma 1.5.2.8, we estimate this further by

$$\le \sum_{\mathfrak{u}\in\mathfrak{U}_1(k,n,j)} D^{1-\kappa Z(n+1)} \mu(\mathcal{I}(\mathfrak{u}))$$

$$\le 2^{100a^2+9a+1-j} \sum_{\mathfrak{m}\in\mathfrak{M}(k,n)} D^{-\kappa Z(n+1)} \mu(\mathcal{I}(\mathfrak{m})).$$

Using Lemma 1.5.2.7, we estimate this by

$$\le 2^{100a^2+9a+1-j} D^{-\kappa Z(n+1)} 2^{n+k+3} \mu(G).$$

Now we estimate G_3 defined in `defineg3` by

$$\mu(G_3) \le \sum_{k\ge 0} \sum_{n\ge k} \sum_{0\le j\le 2n+3} \mu(\bigcup_{\mathfrak{p}\in\mathfrak{L}_4(k,n,j)} \mathcal{I}(\mathfrak{p}))$$

$$\le \sum_{k\ge 0} \sum_{n\ge k} \sum_{0\le j\le 2n+3} 2^{100a^2+9a+3+n+k-j} D^{-\kappa Z(n+1)} \mu(G).$$

Summing geometric series, using that $D^{\kappa Z} \geq 8$ by `defineD`, `definekappa` and `defineZ`, we estimate this by

$$\leq \sum_{k\geq 0} \sum_{n\geq k} 2^{100a^2+9a+4+n+k} D^{-\kappa Z(n+1)} \mu(G)$$

$$= \sum_{k\geq 0} 2^{100a^2+9a+4+2k} D^{-\kappa Z(k+1)} \sum_{n\geq k} 2^{n-k} D^{-\kappa Z(n-k)} \mu(G)$$

$$\leq \sum_{k\geq 0} 2^{100a^2+9a+5+2k} D^{-\kappa Z(k+1)} \mu(G)$$

$$\leq 2^{100a^2+9a+6} D^{-\kappa Z} \mu(G).$$

Using $D = 2^{100a^2}$ and $a \geq 4$ and $\kappa Z \geq 2$ by `defineD` and `definekappa` proves the lemma.

Proof for Lemma 1.5.1.1

Adding up the bounds in Lemmas Lemma 1.5.2.1, Lemma 1.5.2.6, and Lemma 1.5.2.10 proves Lemma 1.5.1.1.

Auxiliary lemmas

Before proving Lemma 1.5.1.2 and Lemma 1.5.1.3, we collect some useful properties of $\lesssim$.

Lemma 1.5.3.1

If $n\mathfrak{p} \lesssim m\mathfrak{p}'$ and $n' \geq n$ and $m \geq m'$ then $n'\mathfrak{p} \lesssim m'\mathfrak{p}'$.

Proof for Lemma 1.5.3.1

Proof. This follows immediately from the definition `wiggleorder` of $\lesssim$ and the two inclusions $B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), n) \subset B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), n')$ and $B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), m') \subset B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), m)$.

Lemma 1.5.3.2

Let $n, m \geq 1$ and $k > 0$. If $\mathfrak{p}, \mathfrak{p}' \in \mathfrak{P}$ with $\mathcal{I}(\mathfrak{p}) \neq \mathcal{I}(\mathfrak{p}')$ and

$$n\mathfrak{p} \lesssim k\mathfrak{p}'$$

then

$$(n + 2^{-95a}m)\mathfrak{p} \lesssim m\mathfrak{p}'.$$

Proof for Lemma 1.5.3.2

Proof. The assumption `eq-wiggle1` together with the definition `wiggleorder` of $\lesssim$ implies that $\mathcal{I}(\mathfrak{p}) \subsetneq \mathcal{I}(\mathfrak{p}')$. Let $\vartheta \in B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), m)$. Then we have by the triangle inequality

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \vartheta) \leq d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{p}')) + d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}'), \vartheta).$$

The first summand is bounded by n since

$$\mathcal{Q}(\mathfrak{p}') \in B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), k) \subset B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), n),$$

using `wiggleorder`. For the second summand we use Lemma 1.2.1.2 to show that the sum is estimated by

$$n + 2^{-95a} d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \vartheta) < n + 2^{-95a} m.$$

Thus $B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), m) \subset B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), n+2^{-95a}m)$. Combined with $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{p}')$, this yields `eq-wiggle2`.

Lemma 1.5.3.3

The following implications hold for all $\mathfrak{q}, \mathfrak{q}' \in \mathfrak{P}$:

$$\mathfrak{q} \leq \mathfrak{q}' \text{ and } \lambda \geq 1.1 \implies \lambda\mathfrak{q} \lesssim \lambda\mathfrak{q}',$$

$$10\mathfrak{q} \lesssim \mathfrak{q}' \text{ and } \mathcal{I}(\mathfrak{q}) \neq \mathcal{I}(\mathfrak{q}') \implies 100\mathfrak{q} \lesssim 100\mathfrak{q}',$$

$$2\mathfrak{q} \lesssim \mathfrak{q}' \text{ and } \mathcal{I}(\mathfrak{q}) \neq \mathcal{I}(\mathfrak{q}') \implies 4\mathfrak{q} \lesssim 500\mathfrak{q}'.$$

Proof for Lemma 1.5.3.3

Proof. `eq-sc2` and `eq-sc3` are easy consequences of Lemma 1.5.3.1, Lemma 1.5.3.2 and the fact that $a \geq 4$. For `eq-sc1`, if $\mathcal{I}(\mathfrak{q}) = \mathcal{I}(\mathfrak{q}')$ then we get $\mathfrak{q} = \mathfrak{q}'$ by `eq-dis-freq-cover` and `straightorder`. If $\mathcal{I}(\mathfrak{q}) \neq \mathcal{I}(\mathfrak{q}')$, then from `straightorder`, `wiggleorder` and `eq-freq-comp-ball` it follows that $\mathfrak{q} \lesssim 0.2\mathfrak{q}'$, and `eq-sc1` follows from an easy calculation using Lemma 1.5.3.2.

We call a collection $\mathfrak{A}$ of tiles convex if

$$\mathfrak{p} \leq \mathfrak{p}' \leq \mathfrak{p}'' \text{ and } \mathfrak{p}, \mathfrak{p}'' \in \mathfrak{A} \implies \mathfrak{p}' \in \mathfrak{A}.$$

Lemma 1.5.3.4

For each k, the collection $\mathfrak{P}(k)$ is convex.

Proof for Lemma 1.5.3.4

Proof. Suppose that $\mathfrak{p} \le \mathfrak{p}' \le \mathfrak{p}''$ and $\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{P}(k)$. By `eq-defPk` we have $\mathcal{I}(\mathfrak{p}), \mathcal{I}(\mathfrak{p}'') \in \mathcal{C}(G, k)$, so there exists by `muhj1` some $J \in \mathcal{D}$ with

$$\mathcal{I}(\mathfrak{p}') \subset \mathcal{I}(\mathfrak{p}'') \subset J$$

and $\mu(G \cap J) > 2^{-k-1}\mu(J)$. Thus `muhj1` holds for $\mathcal{I}(\mathfrak{p}')$. On the other hand, by `muhj2`, there exists no $J \in \mathcal{D}$ with $\mathcal{I}(\mathfrak{p}) \subset J$ and $\mu(G \cap J) > 2^{-k}\mu(J)$. Since $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{p}')$, this implies that `muhj2` holds for $\mathcal{I}(\mathfrak{p}')$. Hence $\mathcal{I}(\mathfrak{p}') \in \mathcal{C}(G, k)$, and therefore by `eq-defPk` $\mathfrak{p}' \in \mathfrak{P}(k)$.

Lemma 1.5.3.5

For each k, n, the collection $\mathfrak{C}(k, n)$ is convex.

Proof for Lemma 1.5.3.5

Proof. Let $\mathfrak{p} \le \mathfrak{p}' \le \mathfrak{p}''$ with $\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{C}(k, n)$. Then, in particular, $\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{P}(k)$, so, by Lemma 1.5.3.4, $\mathfrak{p}' \in \mathfrak{P}(k)$. Next, we show that if $\mathfrak{q} \le \mathfrak{q}' \in \mathfrak{P}(k)$ then $\mathrm{dens}'_k(\{\mathfrak{q}\}) \ge \mathrm{dens}'_k(\{\mathfrak{q}'\})$. If $\mathfrak{p} \in \mathfrak{P}(k)$ and $\lambda \ge 2$ with $\lambda\mathfrak{q}' \lesssim \lambda\mathfrak{p}$, then it follows from $\mathfrak{q} \le \mathfrak{q}'$, `eq-sc1` of Lemma 1.5.3.3 and transitivity of $\lesssim$ that $\lambda\mathfrak{q} \lesssim \lambda\mathfrak{p}$. Thus the supremum in the definition `eq-densdef` of $\mathrm{dens}'_k(\{\mathfrak{q}\})$ is over a superset of the set the supremum in the definition of $\mathrm{dens}'_k(\{\mathfrak{q}'\})$ is taken over, which shows $\mathrm{dens}'_k(\{\mathfrak{q}\}) \ge \mathrm{dens}'_k(\{\mathfrak{q}'\})$. From $\mathfrak{p}' \le \mathfrak{p}''$, $\mathfrak{p}'' \in \mathfrak{C}(k, n)$ and `def-cnk` it then follows that

$$2^{4a}2^{-n} < \mathrm{dens}'_k(\{\mathfrak{p}''\}) \le \mathrm{dens}'_k(\{\mathfrak{p}'\}).$$

Similarly, it follows from $\mathfrak{p} \le \mathfrak{p}'$, $\mathfrak{p} \in \mathfrak{C}(k, n)$ and `def-cnk` that

$$\mathrm{dens}'_k(\{\mathfrak{p}'\}) \le \mathrm{dens}'_k(\{\mathfrak{p}\}) \le 2^{4a}2^{-n+1}.$$

Thus $\mathfrak{p}' \in \mathfrak{C}(k, n)$.

Lemma 1.5.3.6

For each k, n, j, the collection $\mathfrak{C}_1(k, n, j)$ is convex.

Proof for Lemma 1.5.3.6

Proof. Let $\mathfrak{p} \le \mathfrak{p}' \le \mathfrak{p}''$ with $\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{C}_1(k, n, j)$. By Lemma 1.5.3.5 and the inclusion $\mathfrak{C}_1(k, n, j) \subset \mathfrak{C}(k, n)$, which holds by definition `defcnkj`, we have $\mathfrak{p}' \in \mathfrak{C}(k, n)$. By `eq-sc1` and transitivity of $\lesssim$ we have that $\mathfrak{q} \le \mathfrak{q}'$ and $100\mathfrak{q}' \lesssim \mathfrak{m}$ imply $100\mathfrak{q} \lesssim \mathfrak{m}$. So, by `defbfp`, $\mathfrak{B}(\mathfrak{p}'') \subset \mathfrak{B}(\mathfrak{p}') \subset \mathfrak{B}(\mathfrak{p})$. Consequently, by `defcnkj`

$$2^j \le |\mathfrak{B}(\mathfrak{p}'')| \le |\mathfrak{B}(\mathfrak{p}')| \le |\mathfrak{B}(\mathfrak{p})| < 2^{j+1},$$

thus $\mathfrak{p}' \in \mathfrak{C}_1(k, n, j)$.

Lemma 1.5.3.7

For each k, n, j, the collection $\mathfrak{C}_2(k,n,j)$ is convex.

Proof for Lemma 1.5.3.7

Proof. Let $\mathfrak{p} \le \mathfrak{p}' \le \mathfrak{p}''$ with $\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{C}_2(k,n,j)$. By `eq-C2-def`, we have $\mathfrak{C}_2(k,n,j) \subset \mathfrak{C}_1(k,n,j)$. Combined with Lemma 1.5.3.6, it follows that $\mathfrak{p}' \in \mathfrak{C}_1(k,n,j)$. If $\mathfrak{p} = \mathfrak{p}'$ the statement is trivially true, otherwise suppose that $\mathfrak{p}' \notin \mathfrak{C}_2(k,n,j)$. By `eq-C2-def`, this implies that there exists $0 \le l' \le Z(n+1)$ with $\mathfrak{p}' \in \mathfrak{L}_1(k,n,j,l')$. By the definition `eq-L1-def` of $\mathfrak{L}_1(k,n,j,l')$, this implies that $\mathfrak{p}'$ is minimal with respect to $\le$ in $\mathfrak{C}_1(k,n,j) \setminus \bigcup_{l<l'} \mathfrak{L}_1(k,n,j,l)$. Since $\mathfrak{p} \le \mathfrak{p}'$ and $\mathfrak{p} \in \mathfrak{C}_2(k,n,j)$, $\mathfrak{p} = \mathfrak{p}'$, a contradiction.

Lemma 1.5.3.8

For each k, n, j, the collection $\mathfrak{C}_3(k,n,j)$ is convex.

Proof for Lemma 1.5.3.8

Proof. Let $\mathfrak{p} \le \mathfrak{p}' \le \mathfrak{p}''$ with $\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{C}_3(k,n,j)$. By `eq-C3-def` and Lemma 1.5.3.7 it follows that $\mathfrak{p}' \in \mathfrak{C}_2(k,n,j)$. By `eq-C3-def` and `eq-L2-def`, there exists $\mathfrak{u} \in \mathfrak{U}_1(k,n,j)$ with $2\mathfrak{p}'' \lesssim \mathfrak{u}$ and $\mathcal{I}(\mathfrak{p}'') \subsetneq \mathcal{I}(\mathfrak{u})$. From $\mathfrak{p}' \le \mathfrak{p}''$, `eq-sc1` and transitivity of $\lesssim$ we then have $2\mathfrak{p}' \lesssim \mathfrak{u}$ and $\mathcal{I}(\mathfrak{p}') \subsetneq \mathcal{I}(\mathfrak{u})$, so $\mathfrak{p}' \in \mathfrak{C}_3(k,n,j)$.

Lemma 1.5.3.9

For each k, n, j, the collection $\mathfrak{C}_4(k,n,j)$ is convex.

Proof for Lemma 1.5.3.9

Proof. The proof is entirely analogous to Lemma 1.5.3.7, substituting $\mathfrak{C}_4$ for $\mathfrak{C}_2$, $\mathfrak{C}_3$ for $\mathfrak{C}_1$ and $\mathfrak{p}' \le \mathfrak{p}''$ for $\mathfrak{p} \le \mathfrak{p}'$.

Lemma 1.5.3.10

For each k, n, j, the collection $\mathfrak{C}_5(k,n,j)$ is convex.

Proof for Lemma 1.5.3.10

Proof. Let $\mathfrak{p} \le \mathfrak{p}' \le \mathfrak{p}''$ with $\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{C}_5(k,n,j)$. Then $\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{C}_4(k,n,j)$ by `defc5`, and thus by Lemma 1.5.3.9 $\mathfrak{p}' \in \mathfrak{C}_4(k,n,j)$. It suffices to show that if $\mathfrak{p}' \in \mathfrak{L}_4(k,n,j)$ then $\mathfrak{p} \in \mathfrak{L}_4(k,n,j)$ by contraposition; this is true by `eq-L4-def` and $\mathfrak{p} \le \mathfrak{p}'$.

Lemma 1.5.3.11

We have for every $k \geq 0$ and $\mathfrak{P}' \subset \mathfrak{P}(k)$

$$\mathrm{dens}_1(\mathfrak{P}') \leq \mathrm{dens}'_k(\mathfrak{P}').$$

Proof for Lemma 1.5.3.11

Proof. It suffices to show that for all $\mathfrak{p}' \in \mathfrak{P}'$ and $\lambda \geq 2$ and $\mathfrak{p} \in \mathfrak{P}(\mathfrak{P}')$ with $\lambda\mathfrak{p}' \lesssim \lambda\mathfrak{p}$ we have

$$\frac{\mu(E_2(\lambda, \mathfrak{p}))}{\mu(\mathcal{I}(\mathfrak{p}))} \leq \sup_{\mathfrak{p}'' \in \mathfrak{P}(k): \lambda\mathfrak{p}' \lesssim \lambda\mathfrak{p}''} \frac{\mu(E_2(\lambda, \mathfrak{p}''))}{\mu(\mathcal{I}(\mathfrak{p}''))}.$$

Let such $\mathfrak{p}'$, λ, $\mathfrak{p}$ be given. It suffices to show that $\mathfrak{p} \in \mathfrak{P}(k)$, that is, it satisfies `muhj1` and `muhj2`. We show `muhj1`. As $\mathfrak{p} \in \mathfrak{P}(\mathfrak{P}')$, there exists $\mathfrak{p}'' \in \mathfrak{P}'$ with $\mathcal{I}(\mathfrak{p}') \subset \mathcal{I}(\mathfrak{p}'')$. By assumption on $\mathfrak{P}'$, we have $\mathfrak{p}'' \in \mathfrak{P}(k)$ and there exists $J \in \mathcal{D}$ with $\mathcal{I}(\mathfrak{p}'') \subset J$ and

$$\mu(G \cap J) > 2^{-k-1}\mu(J).$$

Then also $\mathcal{I}(\mathfrak{p}') \subset J$, which proves `muhj1` for $\mathfrak{p}$. We show `muhj2`. Assume to get a contradiction that there exists $J \in \mathcal{D}$ with $\mathcal{I}(\mathfrak{p}) \subset J$ and

$$\mu(G \cap J) > 2^{-k}\mu(J).$$

As $\lambda\mathfrak{p}' \lesssim \lambda\mathfrak{p}$, we have $\mathcal{I}(\mathfrak{p}') \subset \mathcal{I}(\mathfrak{p})$, and therefore $\mathcal{I}(\mathfrak{p}') \subset J$. This contradicts $\mathfrak{p}' \in \mathfrak{P}' \subset \mathfrak{P}(k)$. This proves `muhj2` for $\mathfrak{p}$.

Lemma 1.5.3.12

For each set $\mathfrak{A} \subset \mathfrak{C}(k, n)$, we have

$$\mathrm{dens}_1(\mathfrak{A}) \leq 2^{4a}2^{-n+1}.$$

Proof for Lemma 1.5.3.12

Proof. We have by Lemma 1.5.3.11 that $\mathrm{dens}_1(\mathfrak{A}) \leq \mathrm{dens}'_k(\mathfrak{A})$. Since $\mathfrak{A} \subset \mathfrak{C}(k, n)$, it follows from monotonicity of suprema and the definition `eq-densdef` that $\mathrm{dens}'_k(\mathfrak{A}) \leq \mathrm{dens}'_k(\mathfrak{C}(k, n))$. By `eq-densdef` and `def-cnk`, we have

$$\mathrm{dens}'_k(\mathfrak{C}(k, n)) = \sup_{\mathfrak{p} \in \mathfrak{C}(k,n)} \mathrm{dens}'_k(\{\mathfrak{p}\}) \leq 2^{4a}2^{-n+1}.$$

Proof of the Forest Union Lemma

Fix $k, n, j \geq 0$. Define

$$\mathfrak{C}_6(k,n,j)$$

to be the set of all tiles $\mathfrak{p} \in \mathfrak{C}_5(k,n,j)$ such that $\mathcal{I}(\mathfrak{p}) \not\subset G'$. The following chain of lemmas establishes that the set $\mathfrak{C}_6(k,n,j)$ can be written as a union of a small number of n-forests.

For $\mathfrak{u} \in \mathfrak{U}_1(k,n,j)$, define

$$\mathfrak{T}_1(\mathfrak{u}) := \{\mathfrak{p} \in \mathfrak{C}_1(k,n,j) \ : \mathcal{I}(\mathfrak{p}) \neq \mathcal{I}(\mathfrak{u}),\ 2\mathfrak{p} \lesssim \mathfrak{u}\}.$$

Define

$$\mathfrak{U}_2(k,n,j) := \{\mathfrak{u} \in \mathfrak{U}_1(k,n,j) \ : \ \mathfrak{T}_1(\mathfrak{u}) \cap \mathfrak{C}_6(k,n,j) \neq \emptyset\}.$$

Define a relation $\sim$ on $\mathfrak{U}_2(k,n,j)$ by setting $\mathfrak{u} \sim \mathfrak{u}'$ for $\mathfrak{u}, \mathfrak{u}' \in \mathfrak{U}_2(k,n,j)$ if $\mathfrak{u} = \mathfrak{u}'$ or there exists $\mathfrak{p}$ in $\mathfrak{T}_1(\mathfrak{u})$ with $10\mathfrak{p} \lesssim \mathfrak{u}'$.

Lemma 1.5.4.1

If $\mathfrak{u} \sim \mathfrak{u}'$, then $\mathcal{I}(u) = \mathcal{I}(u')$ and

$$B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), 100) \cap B_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), 100) \neq \emptyset.$$

Proof for Lemma 1.5.4.1

Proof. Let $\mathfrak{u}, \mathfrak{u}' \in \mathfrak{U}_2(k,n,j)$ with $\mathfrak{u} \sim \mathfrak{u}'$. If $\mathfrak{u} = \mathfrak{u}'$ then the conclusion of the Lemma clearly holds. Else, there exists $\mathfrak{p} \in \mathfrak{C}_1(k,n,j)$ such that $\mathcal{I}(\mathfrak{p}) \neq \mathcal{I}(\mathfrak{u})$ and $2\mathfrak{p} \lesssim \mathfrak{u}$ and $10\mathfrak{p} \lesssim \mathfrak{u}'$. Using Lemma 1.5.3.1 and `eq-sc2` of Lemma 1.5.3.3, we deduce that

$$100\mathfrak{p} \lesssim 100\mathfrak{u}\,, \qquad 100\mathfrak{p} \lesssim 100\mathfrak{u}'.$$

Now suppose that $B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), 100) \cap B_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), 100) = \emptyset$. Then we have $\mathfrak{B}(\mathfrak{u}) \cap \mathfrak{B}(\mathfrak{u}') = \emptyset$, by the definition `defbfp` of $\mathfrak{B}$ and the definition `wiggleorder` of $\lesssim$, but also $\mathfrak{B}(\mathfrak{u}) \subset \mathfrak{B}(\mathfrak{p})$ and $\mathfrak{B}(\mathfrak{u}') \subset \mathfrak{B}(\mathfrak{p})$, by `defbfp`, `wiggleorder` and the previous estimate. Hence

$$|\mathfrak{B}(\mathfrak{p})| \geq |\mathfrak{B}(\mathfrak{u})| + |\mathfrak{B}(\mathfrak{u}')| \geq 2^j + 2^j = 2^{j+1},$$

which contradicts $\mathfrak{p} \in \mathfrak{C}_1(k,n,j)$. Therefore we must have $B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), 100) \cap B_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), 100) \neq \emptyset$.It follows from $2\mathfrak{p} \lesssim \mathfrak{u}$ and $10\mathfrak{p} \lesssim \mathfrak{u}'$ that $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{u})$ and $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{u}')$. By `dyadicproperty`, it follows that $\mathcal{I}(\mathfrak{u})$ and $\mathcal{I}(\mathfrak{u}')$ are nested. Combining this with the conclusion of the last paragraph and definition `defunkj` of $\mathfrak{U}_1(k,n,j)$, we obtain that $\mathcal{I}(\mathfrak{u}) = \mathcal{I}(\mathfrak{u}')$.

Lemma 1.5.4.2

For each k, n, j, the relation $\sim$ on $\mathfrak{U}_2(k, n, j)$ is an equivalence relation.

Proof for Lemma 1.5.4.2

Proof. Reflexivity holds by definition. For transitivity, suppose that

$$\mathfrak{u}, \mathfrak{u}', \mathfrak{u}'' \in \mathfrak{U}_1(k, n, j)$$

and $\mathfrak{u} \sim \mathfrak{u}'$, $\mathfrak{u}' \sim \mathfrak{u}''$. By Lemma 1.5.4.1, it follows that $\mathcal{I}(\mathfrak{u}) = \mathcal{I}(\mathfrak{u}') = \mathcal{I}(\mathfrak{u}'')$, that there exists

$$\vartheta \in B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), 100) \cap B_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), 100)$$

and that there exists

$$\theta \in B_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), 100) \cap B_{\mathfrak{u}''}(\mathcal{Q}(\mathfrak{u}''), 100).$$

If $\mathfrak{u} = \mathfrak{u}'$, then $\mathfrak{u} \sim \mathfrak{u}''$ holds by assumption. Else, there exists by the definition of $\sim$ some $\mathfrak{p} \in \mathfrak{T}_1(\mathfrak{u})$ with $10\mathfrak{p} \lesssim \mathfrak{u}'$. Then we have $2\mathfrak{p} \lesssim \mathfrak{u}$ and $\mathfrak{p} \neq \mathfrak{u}$ by definition of $\mathfrak{T}_1(\mathfrak{u})$, so $4\mathfrak{p} \lesssim 500\mathfrak{u}$ by `eq-sc3`. For $q \in B_{\mathfrak{u}''}(\mathcal{Q}(\mathfrak{u}''), 1)$ it follows by the triangle inequality that

$$d_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), q) \le d_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), \vartheta) + d_{\mathfrak{u}}(\vartheta, \mathcal{Q}(\mathfrak{u}')) + d_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}'), \theta) + d_{\mathfrak{u}}(\theta, \mathcal{Q}(\mathfrak{u}'')) + d_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}''), q).$$

Using `defdp` and the fact that $\mathcal{I}(\mathfrak{u}) = \mathcal{I}(\mathfrak{u}') = \mathcal{I}(\mathfrak{u}'')$ this equals

$$d_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), \vartheta) + d_{\mathfrak{u}'}(\vartheta, \mathcal{Q}(\mathfrak{u}')) + d_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), \theta) + d_{\mathfrak{u}''}(\theta, \mathcal{Q}(\mathfrak{u}'')) + d_{\mathfrak{u}''}(\mathcal{Q}(\mathfrak{u}''), q) < 100+100+100+100+1 < 500.$$

Since $4\mathfrak{p} \lesssim 500\mathfrak{u}$, it follows that $d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), q) < 4 < 10$. We have shown that $B_{\mathfrak{u}''}(\mathcal{Q}(\mathfrak{u}''), 1) \subset B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 10)$, combining this with $\mathcal{I}(\mathfrak{u}'') = \mathcal{I}(\mathfrak{u})$ gives $\mathfrak{u} \sim \mathfrak{u}''$. For symmetry, suppose that $\mathfrak{u} \sim \mathfrak{u}'$. By Lemma 1.5.4.1, the two tops have the same spatial cube and their `100`-balls in the frequency metric intersect. If $\mathfrak{u} = \mathfrak{u}'$, there is nothing to prove, so assume $\mathfrak{u} \neq \mathfrak{u}'$. Then some tile in $\mathfrak{T}_1(\mathfrak{u}')$ witnesses the relation, and Lemma 1.5.3.1 together with `eq-sc3` upgrades this to the comparison $10\mathfrak{p} \lesssim 4\mathfrak{p} \lesssim 500\mathfrak{u}'$. For any $qB_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), 1)$, the triangle inequality and the equality of the underlying cubes give a bound $d_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), q) < 500$. Combining this with the previous comparison and the definition of $\lesssim$ shows that $B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), 1) B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 10)$. Since $2\mathfrak{p} \lesssim \mathfrak{u}'$, we also have $\mathcal{I}(\mathfrak{p})\mathcal{I}(\mathfrak{u}') = \mathcal{I}(\mathfrak{u})$, and hence the required comparison proving $\mathfrak{u}' \sim \mathfrak{u}$.

Choose a set $\mathfrak{U}_3(k, n, j)$ of representatives for the equivalence classes of $\sim$ in $\mathfrak{U}_2(k, n, j)$. Define for each $\mathfrak{u} \in \mathfrak{U}_3(k, n, j)$

$$\mathfrak{T}_2(\mathfrak{u}) := \bigcup_{\mathfrak{u} \sim \mathfrak{u}'} \mathfrak{T}_1(\mathfrak{u}') \cap \mathfrak{C}_6(k, n, j).$$

Lemma 1.5.4.3

We have
$$\mathfrak{C}_6(k,n,j) = \bigcup_{\mathfrak{u}\in\mathfrak{U}_3(k,n,j)} \mathfrak{T}_2(\mathfrak{u}).$$

Proof for Lemma 1.5.4.3

Proof. Let $\mathfrak{p} \in \mathfrak{C}_6(k,n,j)$. By `eq-C4-def` and `defc5`, we have $\mathfrak{p} \in \mathfrak{C}_3(k,n,j)$. By `eq-L2-def` and `eq-C3-def`, there exists $\mathfrak{u} \in \mathfrak{U}_1(k,n,j)$ with $2\mathfrak{p} \lesssim \mathfrak{u}$ and $\mathcal{I}(\mathfrak{p}) \neq \mathcal{I}(\mathfrak{u})$, that is, with $\mathfrak{p} \in \mathfrak{T}_1(\mathfrak{u})$. Then $\mathfrak{T}_1(\mathfrak{u})$ is clearly nonempty, so $\mathfrak{u} \in \mathfrak{U}_2(k,n,j)$. By the definition of $\mathfrak{U}_3(k,n,j)$, there exists $\mathfrak{u}' \in \mathfrak{U}_3(k,n,j)$ with $\mathfrak{u} \sim \mathfrak{u}'$. By `definesv`, we have $\mathfrak{p} \in \mathfrak{T}_2(\mathfrak{u}')$.

Lemma 1.5.4.4

For each $\mathfrak{u} \in \mathfrak{U}_3(k,n,j)$, the set $\mathfrak{T}_2(\mathfrak{u})$ satisfies `forest1`.

Proof for Lemma 1.5.4.4

Proof. Let $\mathfrak{p} \in \mathfrak{T}_2(\mathfrak{u})$. By `definesv`, there exists $\mathfrak{u}' \sim \mathfrak{u}$ with $\mathfrak{p} \in \mathfrak{T}_1(\mathfrak{u}')$. Then we have $2\mathfrak{p} \lesssim \mathfrak{u}'$ and $\mathcal{I}(\mathfrak{p}) \neq \mathcal{I}(\mathfrak{u}')$, so by `eq-sc3` $4\mathfrak{p} \lesssim 500\mathfrak{u}'$. Further, by Lemma 1.5.4.1, we have that $\mathcal{I}(\mathfrak{u}') = \mathcal{I}(\mathfrak{u})$ and there exists $\vartheta \in B_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'),100) \cap B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}),100)$. Let $\theta \in B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}),1)$. Using the triangle inequality and the fact that $\mathcal{I}(\mathfrak{u}') = \mathcal{I}(\mathfrak{u})$, we obtain

$$d_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'),\theta) \le d_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'),\vartheta) + d_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}),\vartheta) + d_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}),\theta)$$

$$= d_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'),\vartheta) + d_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}),\vartheta) + d_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}),\theta) < 100 + 100 + 1 < 500.$$

Combining this with $4\mathfrak{p} \lesssim 500\mathfrak{u}'$, we obtain

$$B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}),1) \subset B_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'),500) \subset B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}),4).$$

Together with $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{u}') = \mathcal{I}(\mathfrak{u})$, this gives $4\mathfrak{p} \lesssim \mathfrak{u}$, which is `forest1`.

Lemma 1.5.4.5

For each $\mathfrak{u} \in \mathfrak{U}_3(k,n,j)$, the set $\mathfrak{T}_2(\mathfrak{u})$ satisfies the convexity condition `forest2`.

Proof for Lemma 1.5.4.5

Proof. Let $\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{T}_2(\mathfrak{u})$ and $\mathfrak{p}' \in \mathfrak{P}$ with $\mathfrak{p} \le \mathfrak{p}' \le \mathfrak{p}''$. By `definesv` we have $\mathfrak{p}, \mathfrak{p}'' \in \mathfrak{C}_6(k,n,j) \subset \mathfrak{C}_5(k,n,j)$. By Lemma 1.5.3.10, we have $\mathfrak{p}' \in \mathfrak{C}_5(k,n,j)$. Since $\mathfrak{p} \in \mathfrak{C}_6(k,n,j)$ we have $\mathcal{I}(\mathfrak{p}) \not\subset G'$, so $\mathcal{I}(\mathfrak{p}') \not\subset G'$ and therefore also $\mathfrak{p}' \in \mathfrak{C}_6(k,n,j)$. By `definesv` there exists $\mathfrak{u}' \in \mathfrak{U}_2(k,n,j)$ with $\mathfrak{p}'' \in \mathfrak{T}_1(\mathfrak{u}')$ and hence $2\mathfrak{p}'' \lesssim \mathfrak{u}'$ and

$\mathcal{I}(\mathfrak{p}'') \neq \mathcal{I}(\mathfrak{u}')$. Together this implies $\mathcal{I}(\mathfrak{p}'') \subsetneq \mathcal{I}(\mathfrak{u}')$. With the inclusion $\mathcal{I}(\mathfrak{p}') \subset \mathcal{I}(\mathfrak{p}'')$ from $\mathfrak{p}' \le \mathfrak{p}''$, it follows that $\mathcal{I}(\mathfrak{p}') \subsetneq \mathcal{I}(\mathfrak{u}')$ and hence $\mathcal{I}(\mathfrak{p}') \neq \mathcal{I}(\mathfrak{u}')$. By `eq-sc1` and transitivity of $\lesssim$ we further have $2\mathfrak{p}' \lesssim \mathfrak{u}'$, so $\mathfrak{p}' \in \mathfrak{T}_1(\mathfrak{u}')$. It follows that $\mathfrak{p}' \in \mathfrak{T}_2(\mathfrak{u})$, which shows `forest2`.

Lemma 1.5.4.6

For each $\mathfrak{u}, \mathfrak{u}' \in \mathfrak{U}_3(k,n,j)$ with $\mathfrak{u} \neq \mathfrak{u}'$ and each $\mathfrak{p} \in \mathfrak{T}_2(\mathfrak{u})$ with $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{u}')$ we have

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{u}')) > 2^{Z(n+1)}.$$

Proof for Lemma 1.5.4.6

Proof. By the definition `eq-C2-def` of $\mathfrak{C}_2(k,n,j)$, there exists a tile $\mathfrak{p}' \in \mathfrak{C}_1(k,n,j)$ with $\mathfrak{p}' \le \mathfrak{p}$ and $\mathrm{s}(\mathfrak{p}') \le \mathrm{s}(\mathfrak{p}) - Z(n+1)$. By Lemma 1.2.1.2 we have

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{u}')) \ge 2^{95aZ(n+1)} d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{u}')).$$

By `eq-sc1` we have $2\mathfrak{p}' \lesssim 2\mathfrak{p}$, so by transitivity there exists a tile $\mathfrak{v}$ equivalent to $\mathfrak{u}$ with $2\mathfrak{p}' \lesssim \mathfrak{v}$ and with spatial cube different from that of $\mathfrak{p}'$. Since $\mathfrak{u}$ and $\mathfrak{u}'$ are not equivalent under $\sim$, it follows that $10\mathfrak{p}' \not\lesssim \mathfrak{u}'$, so there is a point q in $B_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), 1)$ but outside $B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), 10)$. Using $\mathfrak{p}' \le \mathfrak{p}$, $\mathcal{I}(\mathfrak{p}') \subset \mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{u}')$, and Lemma 1.2.1.2, one gets a lower bound for $d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{u}'))$ that is strictly greater than 8. Combining this with the first display and the fact that $95a$≥1 gives the claimed lower bound.

Lemma 1.5.4.7

For each $\mathfrak{u} \in \mathfrak{U}_3(k,n,j)$ and each $\mathfrak{p} \in \mathfrak{T}_2(\mathfrak{u})$ we have

$$B(\mathrm{c}(\mathfrak{p}), 8D^{\mathrm{s}(\mathfrak{p})}) \subset \mathcal{I}(\mathfrak{u}).$$

Proof for Lemma 1.5.4.7

Proof. Let $\mathfrak{p} \in \mathfrak{T}_2(\mathfrak{u})$. Then $\mathfrak{p} \in \mathfrak{C}_4(k,n,j)$, hence there exists a chain

$$\mathfrak{p} \le \mathfrak{p}_{Z(n+1)} \le \cdots \le \mathfrak{p}_0$$

of distinct tiles in $\mathfrak{C}_3(n,k,j)$. We pick such a chain and set $\mathfrak{q} = \mathfrak{p}_0$. Then we have from distinctness of the tiles in the chain that $\mathrm{s}(\mathfrak{p}) \le \mathrm{s}(\mathfrak{q}) - Z(n+1)$. By `eq-C3-def` there exists $\mathfrak{u}'' \in \mathfrak{U}_1(k,n,j)$ with $2\mathfrak{q} \lesssim \mathfrak{u}''$ and $\mathrm{s}(\mathfrak{q}) < \mathrm{s}(\mathfrak{u}'')$. Then we have in particular by Lemma 1.5.3.1 that $10\mathfrak{p} \lesssim \mathfrak{u}''$. Let $\mathfrak{u}' \sim \mathfrak{u}$ be such that $\mathfrak{p} \in \mathfrak{T}_1(\mathfrak{u}')$. By the definition of $\sim$, it follows that $\mathfrak{u}' \sim \mathfrak{u}''$. By transitivity of $\sim$, we have $\mathfrak{u} \sim \mathfrak{u}''$. By Lemma 1.5.4.1, we have $\mathrm{s}(\mathfrak{u}'') = \mathrm{s}(\mathfrak{u})$, hence $\mathrm{s}(\mathfrak{q}) < \mathrm{s}(\mathfrak{u})$ and $\mathrm{s}(\mathfrak{p}) \le \mathrm{s}(\mathfrak{q}) - Z(n+1) \le \mathrm{s}(\mathfrak{u}) - Z(n+1) - 1$. Thus, there exists some cube $I \in \mathcal{D}$ with $s(I) = \mathrm{s}(\mathfrak{u}) - Z(n+1) - 1$ and $I \subset \mathcal{I}(\mathfrak{u})$ and $\mathcal{I}(\mathfrak{p}) \subset I$. Since $\mathfrak{p} \in \mathfrak{C}_5(k,n,j)$, we have that $I \notin \mathcal{L}(\mathfrak{u})$, so $B(c(I), 8D^{s(I)}) \subset \mathcal{I}(\mathfrak{u})$. By the triangle inequality, `defineD` and $a \ge 4$, the same then holds for the subcube $\mathcal{I}(\mathfrak{p}) \subset I$.

Lemma 1.5.4.8

It holds for $k \le n$ that

$$\sum_{\mathfrak{u} \in \mathfrak{U}_3(k,n,j)} \mathbf{1}_{\mathcal{I}(\mathfrak{u})} \le (4n+12)2^n.$$

Proof for Lemma 1.5.4.8

Proof. Suppose that a point x is contained in more than $(4n+12)2^n$ cubes $\mathcal{I}(\mathfrak{u})$ with $\mathfrak{u} \in \mathfrak{U}_3(k,n,j)$. Since $\mathfrak{U}_3(k,n,j) \subset \mathfrak{C}_1(k,n,j)$ for each such $\mathfrak{u}$, there exists $\mathfrak{m} \in \mathfrak{M}(k,n)$ such that $100\mathfrak{u} \lesssim \mathfrak{m}$. We fix such an $\mathfrak{m}(\mathfrak{u}) := \mathfrak{m}$ for each $\mathfrak{u}$, and claim that the map $\mathfrak{u} \mapsto \mathfrak{m}(\mathfrak{u})$ is injective. Indeed, assume for $\mathfrak{u} \ne \mathfrak{u}'$ there is $\mathfrak{m} \in \mathfrak{M}(k,n)$ such that $100\mathfrak{u} \lesssim \mathfrak{m}$ and $100\mathfrak{u}' \lesssim \mathfrak{m}$. By `dyadicproperty`, either $\mathcal{I}(\mathfrak{u}) \subset \mathcal{I}(\mathfrak{u}')$ or $\mathcal{I}(\mathfrak{u}') \subset \mathcal{I}(\mathfrak{u})$. By `defunkj`, $B_{\mathfrak{u}}(\mathcal{Q}(\mathfrak{u}), 100) \cap B_{\mathfrak{u}'}(\mathcal{Q}(\mathfrak{u}'), 100) = \emptyset$. This contradicts $\Omega(\mathfrak{m})$ being contained in both sets by `eq-freq-comp-ball`. Thus x is contained in more than $(4n+12)2^n$ cubes $\mathcal{I}(\mathfrak{m})$, $\mathfrak{m} \in \mathfrak{M}(k,n)$. Consequently, we have by `eq-Aoverlap-def` that $x \in A(2n+6,k,n) \subset G_2$. Let $\mathcal{I}(\mathfrak{u})$ be an inclusion minimal cube among the $\mathcal{I}(\mathfrak{u}'), \mathfrak{u}' \in \mathfrak{U}_3(k,n,j)$ with $x \in \mathcal{I}(\mathfrak{u})$. By the dyadic property `dyadicproperty`, we have $\mathcal{I}(\mathfrak{u}) \subset \mathcal{I}(\mathfrak{u}')$ for all cubes $\mathcal{I}(\mathfrak{u}')$ containing x. Thus

$$\mathcal{I}(\mathfrak{u}) \subset \{y : \sum_{\mathfrak{u} \in \mathfrak{U}_3(k,n,j)} \mathbf{1}_{\mathcal{I}(\mathfrak{u})}(y) > 1 + (4n+12)2^n\} \subset G_2.$$

Thus $\mathfrak{T}_1(\mathfrak{u}) \cap \mathfrak{C}_6(k,n,j) = \emptyset$. This contradicts $\mathfrak{u} \in \mathfrak{U}_2(k,n,j)$.

We now turn to the proof of Lemma 1.5.1.2.

Proof for Lemma 1.5.1.2

Proof of Lemma 1.5.1.2. We first fix k, n, j. By `definetp` and `defineep`, we have that $\mathbf{1}_{\mathcal{I}(\mathfrak{p})} T_{\mathfrak{p}} f(x) = T_{\mathfrak{p}} f(x)$ and hence $\mathbf{1}_{G \setminus G'} T_{\mathfrak{p}} f(x) = 0$ for all $\mathfrak{p} \in \mathfrak{C}_5(k,n,j) \setminus \mathfrak{C}_6(k,n,j)$. Thus it suffices to estimate the contribution of the sets $\mathfrak{C}_6(k,n,j)$. By Lemma 1.5.4.8, we can decompose $\mathfrak{U}_3(k,n,j)$ as a disjoint union of at most $4n+12$ collections $\mathfrak{U}_4(k,n,j,l)$, $1 \le l \le 4n+12$, each satisfying

$$\sum_{\mathfrak{u} \in \mathfrak{U}_4(k,n,j,l)} \mathbf{1}_{\mathcal{I}(\mathfrak{u})} \le 2^n.$$

By Lemmas Lemma 1.5.4.4, Lemma 1.5.4.5, Lemma 1.5.4.6, Lemma 1.5.4.7, and Lemma 1.5.3.12, the pairs

$$(\mathfrak{U}_4(k,n,j,l), \mathfrak{T}_2|_{\mathfrak{U}_4(k,n,j,l)})$$

are n-forests for each k, n, j, l, and by Lemma 1.5.4.3, we have

$$\mathfrak{C}_6(k,n,j) = \bigcup_{l=1}^{4n+12} \bigcup_{\mathfrak{u} \in \mathfrak{U}_4(k,n,j,l)} \mathfrak{T}_2(\mathfrak{u}).$$

Since $\mathcal{I}(\mathfrak{p}) \not\subset G_1$ for all $\mathfrak{p} \in \mathfrak{C}_6(k,n,j)$, we have $\mathfrak{C}_6(k,n,j) \cap \mathfrak{P}_{F,G} = \emptyset$ and hence

$$\mathrm{dens}_2(\bigcup_{\mathfrak{u} \in \mathfrak{U}_4(k,n,j,l)} \mathfrak{T}_2(\mathfrak{u})) \le 2^{2a+5} \frac{\mu(F)}{\mu(G)}.$$

Using the triangle inequality according to the splitting by k, n, j and l in `disclesssiml` and applying Theorem 1.2.4 to each term, we obtain the estimate

$$\sum_{k \ge 0} \sum_{n \ge k} (2n+3)(4n+12) 2^{440a^3} 2^{-(1-\frac{1}{q})n} (2^{2a+5} \frac{\mu(F)}{\mu(G)})^{\frac{1}{q}-\frac{1}{2}} \|f\|_2 \|\mathbf{1}_{G \setminus G'}\|_2$$

for the left hand side of `disclesssiml`. Since $|f| \le \mathbf{1}_F$, we have $\|f\|_2 \le \mu(F)^{1/2}$, and we have $\|\mathbf{1}_{G \setminus G'}\|_2 \le \mu(G)^{1/2}$. We get a bound

$$2^{440a^3} \mu(F)^{\frac{1}{q}} \mu(G)^{1-\frac{1}{q}} 2^{2a+5} \sum_{k \ge 0} \sum_{n \ge k} (2n+3)(4n+12) 2^{-(1-\frac{1}{q})n}.$$

Interchanging the order of summation, the last factor equals

$$2^{2a+5} \sum_{n \ge 0} (2n+3)(4n+12)(n+1) 2^{-\frac{q-1}{q}n}.$$

Up to an explicit constant, the sum is bounded by $\sum n^3 2^{-\frac{q-1}{q}n}$, which is at most some constant times $1/(q-1)^4$ by comparing to an integral. Since $a \ge 4$, this is overall bounded by $2^{a^3}/(q-1)^4$, which completes the proof of the lemma.

Proof of the Forest Complement Lemma

Define $\mathfrak{P}_{G \setminus G'}$ to be the set of all $\mathfrak{p} \in \mathfrak{P}$ such that $\mu(\mathcal{I}(\mathfrak{p}) \cap (G \setminus G')) > 0$.

Lemma 1.5.5.1

We have that

$$\mathfrak{P}_2 \cap \mathfrak{P}_{G \setminus G'} = \bigcup_{k \ge 0} \bigcup_{n \ge k} \mathfrak{L}_0(k,n) \cap \mathfrak{P}_{G \setminus G'} \cup \bigcup_{k \ge 0} \bigcup_{n \ge k} \bigcup_{0 \le j \le 2n+3} \mathfrak{L}_2(k,n,j) \cap \mathfrak{P}_{G \setminus G'} \cup \bigcup_{k \ge 0} \bigcup_{n \ge k} \bigcup_{0 \le j \le 2n+3} \bigcup_{0 \le l \le Z(n+1)}$$

Proof for Lemma 1.5.5.1

Proof. Let $\mathfrak{p} \in \mathfrak{P}_2 \cap \mathfrak{P}_{G\setminus G'}$. Clearly, for every cube $J = \mathcal{I}(\mathfrak{p})$ with $\mathfrak{p} \in \mathfrak{P}_{G\setminus G'}$ there exists some $k \geq 0$ such that `muhj1` holds, and for no cube $J \in \mathcal{D}$ and no $k < 0$ does `muhj2` hold. Thus $\mathfrak{p} \in \mathfrak{P}(k)$ for some $k \geq 0$. Next, since $E_2(\lambda, \mathfrak{p}') \subset \mathcal{I}(\mathfrak{p}') \cap G$ for every $\lambda \geq 2$ and every tile $\mathfrak{p}' \in \mathfrak{P}(k)$ with $\lambda\mathfrak{p} \lesssim \lambda\mathfrak{p}'$, it follows from `muhj2` that $\mu(E_2(\lambda, \mathfrak{p}')) \leq 2^{-k}\mu(\mathcal{I}(\mathfrak{p}'))$ for every such $\mathfrak{p}'$, so $\operatorname{dens}'_k(\{\mathfrak{p}\}) \leq 2^{-k}$. Combining this with $a \geq 0$, it follows from `def-cnk` that there exists $n \geq k$ with $\mathfrak{p} \in \mathfrak{C}(k, n)$. Since $\mathfrak{p} \in \mathfrak{P}_{G\setminus G'}$, we have in particular $\mathcal{I}(\mathfrak{p}) \not\subset A(2n+6, k, n)$, so there exist at most $1+(4n+12)2^n < 2^{2n+4}$ tiles $\mathfrak{m} \in \mathfrak{M}(k, n)$ with $\mathfrak{p} \leq \mathfrak{m}$. It follows that $\mathfrak{p} \in \mathfrak{L}_0(k, n)$ or $\mathfrak{p} \in \mathfrak{C}_1(k, n, j)$ for some $1 \leq j \leq 2n + 3$. In the former case we are done, in the latter case the equality to be shown follows from the definitions of the collections $\mathfrak{C}_i$ and $\mathfrak{L}_i$.

Lemma 1.5.5.2

We have that

$$\mathfrak{L}_0(k, n) = \dot{\bigcup_{0\leq l<n}} \mathfrak{L}_0(k, n, l),$$

where each $\mathfrak{L}_0(k, n, l)$ is an antichain.

Proof for Lemma 1.5.5.2

Proof. It suffices to show that $\mathfrak{L}_0(k, n)$ contains no chain of length $n + 1$. Suppose that we had such a chain $\mathfrak{p}_0 \leq \mathfrak{p}_1 \leq \cdots \leq \mathfrak{p}_n$ with $\mathfrak{p}_i \neq \mathfrak{p}_{i+1}$ for $i = 0, \ldots, n-1$. By `def-cnk`, we have that $\operatorname{dens}'_k(\{\mathfrak{p}_n\}) > 2^{-n}$. Thus, by `eq-densdef`, there exists $\mathfrak{p}' \in \mathfrak{P}(k)$ and $\lambda \geq 2$ with $\lambda\mathfrak{p}_n \leq \lambda\mathfrak{p}'$ and $\frac{\mu(E_2(\lambda,\mathfrak{p}'))}{\mu(\mathcal{I}(\mathfrak{p}'))} > \lambda^a 2^{4a} 2^{-n}$. Let $\mathfrak{O}$ be the set of all $\mathfrak{p}'' \in \mathfrak{P}(k)$ such that $\mathcal{I}(\mathfrak{p}'') = \mathcal{I}(\mathfrak{p}')$ and $B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \lambda) \cap \Omega(\mathfrak{p}'') \neq \emptyset$. The balls $B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}''), 0.2)$, $\mathfrak{p}'' \in \mathfrak{O}$ are disjoint by `eq-freq-comp-ball`, and by the triangle inequality contained in $B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \lambda + 1.2)$. By `thirddb`, this ball can be covered with at most $2^{4a}\lambda^a$ many $d_{\mathfrak{p}'}$-balls of radius 0.2, so $|\mathfrak{O}| \leq 2^{4a}\lambda^a$. By `definee1` and `definee2` we have $E_2(\lambda, \mathfrak{p}') \subset \bigcup_{\mathfrak{p}''\in\mathfrak{O}} E_1(\mathfrak{p}'')$, thus $2^{4a}\lambda^a 2^{-n} < \sum_{\mathfrak{p}''\in\mathfrak{O}} \frac{\mu(E_1(\mathfrak{p}''))}{\mu(\mathcal{I}(\mathfrak{p}''))}$. Hence there exists a tile $\mathfrak{p}'' \in \mathfrak{O}$ with $\mu(E_1(\mathfrak{p}'')) \geq 2^{-n}\mu(\mathcal{I}(\mathfrak{p}'))$. By the definition `mnkmax` of $\mathfrak{M}(k, n)$, there exists a tile $\mathfrak{m} \in \mathfrak{M}(k, n)$ with $\mathfrak{p}' \leq \mathfrak{m}$. From the previous inequality, the inclusion $E_2(\lambda, \mathfrak{p}') \subset \mathcal{I}(\mathfrak{p}')$ and $a \geq 1$ we obtain $2^n \geq 2^{4a}\lambda^a \geq \lambda$. From the triangle inequality and Lemma 1.2.1.2, it follows for all $\vartheta \in B_{\mathfrak{m}}(\mathcal{Q}(\mathfrak{m}), 1)$ that $d_{\mathfrak{p}_0}(\mathcal{Q}(\mathfrak{p}_0), \vartheta)$ is bounded by `100`. Thus, by `straightorder`, $100\mathfrak{p}_0 \lesssim \mathfrak{m}$, a contradiction to $\mathfrak{p}_0 \notin \mathfrak{C}(k, n)$.

Lemma 1.5.5.3

Each of the sets $\mathfrak{L}_2(k,n,j)$ is an antichain.

Proof for Lemma 1.5.5.3

Proof. Suppose that there are $\mathfrak{p}_0, \mathfrak{p}_1 \in \mathfrak{L}_2(k,n,j)$ with $\mathfrak{p}_0 \neq \mathfrak{p}_1$ and $\mathfrak{p}_0 \leq \mathfrak{p}_1$. By Lemma 1.5.3.1 and Lemma 1.5.3.2, it follows that $2\mathfrak{p}_0 \lesssim 200\mathfrak{p}_1$. Since $\mathfrak{L}_2(k,n,j)$ is finite, there exists a maximal $l \geq 1$ such that there exists a chain

$$2\mathfrak{p}_0 \lesssim 200\mathfrak{p}_1 \lesssim \cdots \lesssim 200\mathfrak{p}_l$$

with all $\mathfrak{p}_i$ in $\mathfrak{C}_1(k,n,j)$ and $\mathfrak{p}_i \neq \mathfrak{p}_{i+1}$ for $i = 0, \ldots, l-1$. If we have $\mathfrak{p}_l \in \mathfrak{U}_1(k,n,j)$, then it follows from $2\mathfrak{p}_0 \lesssim 200\mathfrak{p}_l \lesssim \mathfrak{p}_l$ and `eq-L2-def` that $\mathfrak{p}_0 \notin \mathfrak{L}_2(k,n,j)$, a contradiction. Thus, by the definition `defunkj` of $\mathfrak{U}_1(k,n,j)$, there exists $\mathfrak{p}_{l+1} \in \mathfrak{C}_1(k,n,j)$ with $\mathcal{I}(\mathfrak{p}_l) \subsetneq \mathcal{I}(\mathfrak{p}_{l+1})$ and $\vartheta \in B_{\mathfrak{p}_l}(\mathcal{Q}(\mathfrak{p}_l), 100) \cap B_{\mathfrak{p}_{l+1}}(\mathcal{Q}(\mathfrak{p}_{l+1}), 100)$. Using the triangle inequality and Lemma 1.2.1.2, one deduces that $200\mathfrak{p}_l \lesssim 200\mathfrak{p}_{l+1}$. This contradicts maximality of l.

Lemma 1.5.5.4

Each of the sets $\mathfrak{L}_1(k,n,j,l)$ and $\mathfrak{L}_3(k,n,j,l)$ is an antichain.

Proof for Lemma 1.5.5.4

Proof. By its definition `eq-L1-def`, each set $\mathfrak{L}_1(k,n,j,l)$ is a set of minimal elements in some set of tiles with respect to $\leq$. If there were distinct $\mathfrak{p}, \mathfrak{q} \in \mathfrak{L}_1(k,n,j,l)$ with $\mathfrak{p} \leq \mathfrak{q}$, then $\mathfrak{q}$ would not be minimal. Hence such $\mathfrak{p}, \mathfrak{q}$ do not exist. Similarly, by `eq-L3-def`, each set $\mathfrak{L}_3(k,n,j,l)$ is a set of maximal elements in some set of tiles with respect to $\leq$. If there were distinct $\mathfrak{p}, \mathfrak{q} \in \mathfrak{L}_3(k,n,j,l)$ with $\mathfrak{p} \leq \mathfrak{q}$, then $\mathfrak{p}$ would not be maximal.

We now turn to the proof of Lemma 1.5.1.3.

Proof for Lemma 1.5.1.3

Proof of Lemma 1.5.1.3. If $\mathfrak{p} \notin \mathfrak{P}_{G \setminus G'}$, then $\mu(\mathcal{I}(\mathfrak{p}) \cap (G \setminus G')) = 0$. By `definetp` and `defineel`, it follows that $\mathbf{1}_{G \setminus G'} T_{\mathfrak{p}} f(x) = 0$ for almost every x. We thus have, almost everywhere,

$$\mathbf{1}_{G \setminus G'} \sum_{\mathfrak{p} \in \mathfrak{P}_2} T_{\mathfrak{p}} f(x) = \mathbf{1}_{G \setminus G'} \sum_{\mathfrak{p} \in \mathfrak{P}_2 \cap \mathfrak{P}_{G \setminus G'}} T_{\mathfrak{p}} f(x).$$

Let $\mathfrak{L}(k,n)$ denote any of the terms $\mathfrak{L}_i(k,n,j,l) \cap \mathfrak{P}_{G \setminus G'}$ on the right hand side of `eq-fp'-decomposition`, where the indices j, l may be void. Then $\mathfrak{L}(k,n)$ is an antichain, by Lemma 1.5.5.2, Lemma 1.5.5.3, and Lemma 1.5.5.4. Further, we have

$$\mathrm{dens}_1(\mathfrak{L}(k,n)) \leq 2^{4a+1-n}$$

by Lemma 1.5.3.12, and we have

$$\mathrm{dens}_2(\mathfrak{L}(k,n)) \le 2^{2a+5}\frac{\mu(F)}{\mu(G)},$$

since

$$\mathfrak{L}(k,n) \cap \mathfrak{P}_{F,G} \subset \mathfrak{P}_{G\setminus G'} \cap \mathfrak{P}_{F,G} = \emptyset.$$

Applying now the triangle inequality according to the decomposition coming from Lemma 1.5.5.1, and then applying Theorem 1.2.3 to each term, we obtain the estimate

$$\sum_{k\ge 0}\sum_{n\ge k}(n+(2n+4)+2(2n+4)(1+Z(n+1)))2^{117a^3}(q-1)^{-1}(2^{4a+1-n})^{\frac{q-1}{8a^4}}(2^{2a+5}\frac{\mu(F)}{\mu(G)})^{\frac{1}{q}-\frac{1}{2}}\|f\|_2\|\mathbf{1}_{G\setminus G'}\|_2$$

for the left hand side of `forestcompleft`. Because $|f| \le \mathbf{1}_F$, we have $\|f\|_2 \le \mu(F)^{1/2}$, and we have $\|\mathbf{1}_{G\setminus G'}\|_2 \le \mu(G)^{1/2}$. Using this and `defineZ`, we bound

$$2^{118a^3}(q-1)^{-1}\mu(F)^{\frac{1}{q}}\mu(G)^{\frac{1}{q'}}\sum_{k\ge 0}\sum_{n\ge k}n^2 2^{-n\frac{q-1}{8a^4}}.$$

The last sum equals, by changing the order of summation,

$$\sum_{n\ge 0}n^2(n+1)2^{-n\frac{q-1}{8a^4}} \le \frac{2^{2a^3}}{(q-1)^4}.$$

This completes the proof.

1.6 Proof of the Antichain Operator Proposition

Let an antichain $\mathfrak{A}$ and functions f, g as in Theorem 1.2.3 be given. We prove `eq-antiprop` in The density arguments as the geometric mean of two inequalities, each involving one of the two densities. One of these two inequalities will need a careful estimate formulated in Lemma 1.6.1.5 of the TT^* correlation between two tile operators. Lemma 1.6.1.5 will be proven in section 1.6 (p. 65).

The summation of the contributions of these individual correlations will require a geometric Lemma 1.6.1.6 counting the relevant tile pairs. Lemma 1.6.1.6 will be proven in section 1.6 (p. 69).

The density arguments

We begin with the following crucial disjointedness property of the sets $E(\mathfrak{p})$ with $\mathfrak{p} \in \mathfrak{A}$.

Lemma 1.6.1.1

Let $\mathfrak{p}, \mathfrak{p}' \in \mathfrak{A}$. If there exists an $x \in X$ with $x \in E(\mathfrak{p}) \cap E(\mathfrak{p}')$, then $\mathfrak{p} = \mathfrak{p}'$.

Proof for Lemma 1.6.1.1

Proof. Let $\mathfrak{p}, \mathfrak{p}'$ and x be given. Assume without loss of generality that $\mathrm{s}(\mathfrak{p}) \le \mathrm{s}(\mathfrak{p}')$. As we have $x \in E(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{p})$ and $x \in E(\mathfrak{p}') \subset \mathcal{I}(\mathfrak{p}')$ by definition `defineep`, we conclude for $i = 1, 2$ that $Q(x) \in \Omega(\mathfrak{p})$ and $Q(x) \in \Omega(\mathfrak{p}')$. By `eq-freq-dyadic` we have $\Omega(\mathfrak{p}') \subset \Omega(\mathfrak{p})$. By definition `straightorder`, we conclude $\mathfrak{p} \le \mathfrak{p}'$. As $\mathfrak{A}$ is an antichain, we conclude $\mathfrak{p} = \mathfrak{p}'$. This proves the lemma.

Let $\mathcal{B}$ be the collection of balls

$$B(\mathrm{c}(\mathfrak{p}), 8D^{\mathrm{s}(\mathfrak{p})})$$

with $\mathfrak{p} \in \mathfrak{A}$ and recall the definition of $M_{\mathcal{B}}$ from Definition `def-hlm`.
Lemma 1.6.1.2

Let $x \in X$. Then

$$|\sum_{\mathfrak{p} \in \mathfrak{A}} T_{\mathfrak{p}} f(x)| \le 2^{102a^3} M_{\mathcal{B}} f(x).$$

Proof for Lemma 1.6.1.2

Proof. Fix $x \in X$. By Lemma 1.6.1.1, there is at most one $\mathfrak{p} \in \mathfrak{A}$ such that $T_{\mathfrak{p}} f(x)$ is not zero. If there is no such $\mathfrak{p}$, the estimate `hlmbound` follows.Assume there is such a $\mathfrak{p}$. By definition of $T_{\mathfrak{p}}$ we have $x \in E(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{p})$ and by the squeezing property `eq-vol-sp-cube`

$$\rho(x, \mathrm{c}(\mathfrak{p})) \le 4D^{\mathrm{s}(\mathfrak{p})}.$$

Let $y \in X$ with $K_{\mathrm{s}(\mathfrak{p})}(x, y) \neq 0$. By definition `defks` of $K_{\mathrm{s}(\mathfrak{p})}$ we have

$$\frac{1}{4} D^{\mathrm{s}(\mathfrak{p})-1} \le \rho(x, y) \le \frac{1}{2} D^{\mathrm{s}(\mathfrak{p})}.$$

The triangle inequality with `eqtttt0` and `supp-Ks1` implies

$$\rho(\mathrm{c}(\mathfrak{p}), y) < 8D^{\mathrm{s}(\mathfrak{p})}.$$

Using the kernel bound `eqkernel-size` and the lower bound in `supp-Ks` we obtain

$$|K_{\mathrm{s}(\mathfrak{p})}(x, y)| \le \frac{2^{a^3}}{\mu(B(x, \frac{1}{4} D^{\mathrm{s}(\mathfrak{p})-1}))}.$$

Using $D = 2^{100a^2}$ and the doubling property `doublingx` $5 + 100a^2$ times estimates the last display by

$$\frac{2^{5a+101a^3}}{\mu(B(x, 8D^{\mathsf{s}(\mathfrak{p})}))},$$

which, thanks to the closeness of the points x and $\mathsf{c}(\mathfrak{p})$ shown in `eqtttt0`, is in turn bounded by

$$\frac{2^{6a+101a^3}}{\mu(B(\mathsf{c}(\mathfrak{p}), 8D^{\mathsf{s}(\mathfrak{p})}))}.$$

Using that $|e(\vartheta)|$ is bounded by 1 for every $\vartheta \in \Theta$, we estimate with the triangle inequality and the above information

$$|T_{\mathfrak{p}} f(x)| \le \frac{2^{6a+101a^3}}{\mu(B(\mathsf{c}(\mathfrak{p}), 8D^{\mathsf{s}(\mathfrak{p})}))} \int_{\mu(B(\mathsf{c}(\mathfrak{p}), 8D^{\mathsf{s}(\mathfrak{p})}))} |f(y)|\, dy$$

This together with $a \ge 4$ proves the lemma.

Set

$$\tilde{q} = \frac{2q}{1+q}.$$

Since $1 < q \le 2$, we have $1 < \tilde{q} < q \le 2$.
Lemma 1.6.1.3

We have that

$$\left| \int \overline{g(x)} \sum_{\mathfrak{p} \in \mathfrak{A}} T_{\mathfrak{p}} f(x)\, d\mu(x) \right| \le 2^{103a^3} (q-1)^{-1} \operatorname{dens}_2(\mathfrak{A})^{\frac{1}{\tilde{q}} - \frac{1}{2}} \|f\|_2 \|g\|_2.$$

Proof for Lemma 1.6.1.3

Proof. We have $f = \mathbf{1}_F f$. Using Holder's inequality, we obtain for each $x \in B'$ and each $B' \in \mathcal{B}$ using $1 < \tilde{q} \le 2$

$$\frac{1}{\mu(B')} \int_{B'} |f(y)|\, d\mu(y)$$

$$\le \left(\frac{1}{\mu(B')} \int_{B'} |f(y)|^{\frac{2\tilde{q}}{3\tilde{q}-2}}\, d\mu(y) \right)^{\frac{3}{2} - \frac{1}{\tilde{q}}} \left(\frac{1}{\mu(B')} \int_{B'} \mathbf{1}_F(y)\, d\mu(y) \right)^{\frac{1}{\tilde{q}} - \frac{1}{2}}$$

$$\le \left(M_{\mathcal{B}}(|f|^{\frac{2\tilde{q}}{3\tilde{q}-2}})(x) \right)^{\frac{3}{2} - \frac{1}{\tilde{q}}} \operatorname{dens}_2(\mathfrak{A})^{\frac{1}{\tilde{q}} - \frac{1}{2}}.$$

Taking the maximum over all B' containing x, we obtain

$$M_{\mathcal{B}}|f| \le M_{\mathcal{B}, \frac{2\tilde{q}}{3\tilde{q}-2}} |f| \operatorname{dens}_2(\mathfrak{A})^{\frac{1}{\tilde{q}} - \frac{1}{2}}.$$

We have with Theorem 1.2.6

$$\left\|M_{\mathcal{B},\frac{2\tilde{q}}{3\tilde{q}-2}}f\right\|_2 \le 2^{2a}(3\tilde{q}-2)(2\tilde{q}-2)^{-1}\|f\|_2.$$

Using $1<\tilde{q}\le 2$ estimates the last display by

$$2^{2a+2}(\tilde{q}-1)^{-1}\|f\|_2.$$

We obtain with Cauchy-Schwarz and then Lemma 1.6.1.2

$$|\int \overline{g(x)}\sum_{\mathfrak{p}\in\mathfrak{A}} T_{\mathfrak{p}}f(x)\,d\mu(x)|$$

$$\le \|g\|_2\Big\|\sum_{\mathfrak{p}\in\mathfrak{A}} T_{\mathfrak{p}}f\Big\|_2$$

$$\le 2^{102a^3}\|g\|_2\|M_{\mathcal{B}}f\|_2.$$

With `eqttt1` and `eqttt2` we can estimate the last display by

$$\le 2^{102a^3+2a+2}(\tilde{q}-1)^{-1}\|g\|_2\|f\|_2\,\mathrm{dens}_2(\mathfrak{A})^{\frac{1}{\tilde{q}}-\frac{1}{2}}$$

Using $a\ge 4$ and $(\tilde{q}-1)^{-1}=(q+1)/(q-1)\le 3(q-1)^{-1}$ proves the lemma.

Lemma 1.6.1.4

Set $p:=4a^4$. We have

$$\left|\int \overline{g(x)}\sum_{\mathfrak{p}\in\mathfrak{A}} T_{\mathfrak{p}}f(x)\,d\mu(x)\right| \le 2^{117a^3}\,\mathrm{dens}_1(\mathfrak{A})^{\frac{1}{2p}}\|f\|_2\|g\|_2.$$

Proof for Lemma 1.6.1.4

Proof. We write for the expression inside the absolute values on the left-hand side of `eqttt3`

$$\sum_{\mathfrak{p}\in\mathfrak{A}}\iint \overline{g(x)}\mathbf{1}_{E(\mathfrak{p})}(x)K_{\mathrm{s}(\mathfrak{p})}(x,y)e(Q(x)(y)-Q(x)(x))f(y)\,d\mu(y)\,d\mu(x)$$

$$=\int\sum_{\mathfrak{p}\in\mathfrak{A}}\overline{T_{\mathfrak{p}}^*g(y)}f(y)\,d\mu(y)$$

with the adjoint operator

$$T_{\mathfrak{p}}^*g(y)=\int_{E(\mathfrak{p})}\overline{K_{\mathrm{s}(\mathfrak{p})}(x,y)}e(-Q(x)(y)+Q(x)(x))g(x)\,d\mu(x).$$

We have by expanding the square

$$\int \Big|\sum_{\mathfrak{p}\in\mathfrak{A}} T_{\mathfrak{p}}^* g(y)\Big|^2 \, d\mu(y) = \int \left(\sum_{\mathfrak{p}\in\mathfrak{A}} T_{\mathfrak{p}}^* g(y)\right) \left(\sum_{\mathfrak{p}'\in\mathfrak{A}} \overline{T_{\mathfrak{p}'}^* g(y)}\right) d\mu(y)$$

$$\le \sum_{\mathfrak{p}\in\mathfrak{A}} \sum_{\mathfrak{p}'\in\mathfrak{A}} \Big| \int T_{\mathfrak{p}}^* g(y) \overline{T_{\mathfrak{p}'}^* g(y)} \, d\mu(y)\Big|.$$

We split the sum into the terms with $\mathrm{s}(\mathfrak{p}') \le \mathrm{s}(\mathfrak{p})$ and $\mathrm{s}(\mathfrak{p}) < \mathrm{s}(\mathfrak{p}')$. Using the symmetry of each summand, we may switch $\mathfrak{p}$ and $\mathfrak{p}'$ in the second sum. Using further positivity of each summand to replace the condition $\mathrm{s}(\mathfrak{p}') < \mathrm{s}(\mathfrak{p})$ by $\mathrm{s}(\mathfrak{p}') \le \mathrm{s}(\mathfrak{p})$ in the second sum, we estimate `eqtts1` by

$$\le 2 \sum_{\mathfrak{p}\in\mathfrak{A}} \sum_{\mathfrak{p}'\in\mathfrak{A}:\mathrm{s}(\mathfrak{p}')\le\mathrm{s}(\mathfrak{p})} \Big| \int T_{\mathfrak{p}}^* g(y) \overline{T_{\mathfrak{p}'}^* g(y)} \, d\mu(y)\Big|.$$

Define for $\mathfrak{p} \in \mathfrak{P}$

$$B(\mathfrak{p}) := B(\mathrm{c}(\mathfrak{p}), 14D^{\mathrm{s}(\mathfrak{p})})$$

and define

$$\mathfrak{A}(\mathfrak{p}) := \{\mathfrak{p}' \in \mathfrak{A} : \mathrm{s}(\mathfrak{p}') \le \mathrm{s}(\mathfrak{p}) \wedge \mathcal{I}(\mathfrak{p}') \subset B(\mathfrak{p})\}.$$

Note that by the squeezing property `eq-vol-sp-cube` and the doubling property `doublingx` applied 6 times we have

$$\mu(B(\mathfrak{p})) \le 2^{6a}\mu(\mathcal{I}(\mathfrak{p})).$$

Using Lemma 1.6.1.5 and `eqttt4`, we estimate `eqtts2` by

$$\le 2^{232a^3+6a+1} \sum_{\mathfrak{p}\in\mathfrak{A}} \int_{E(\mathfrak{p})} |g|(y) h(\mathfrak{p}) \, d\mu(y)$$

with $h(\mathfrak{p})$ defined as

$$\frac{1}{\mu(B(\mathfrak{p}))} \int \sum_{\mathfrak{p}'\in\mathfrak{A}(\mathfrak{p})} (1 + d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \mathcal{Q}(\mathfrak{p}))^{-1/(2a^2+a^3)} (\mathbf{1}_{E(\mathfrak{p}')}|g|)(y') \, d\mu(y').$$

Note that $p > 4$ since $a \ge 4$. Let p' be the dual exponent of p, satisfying $1/p + 1/p' = 1$. We estimate $h(\mathfrak{p})$ as defined in `def-hp` with Holder using $|g| \le \mathbf{1}_G$ and $E(\mathfrak{p}') \subset B(\mathfrak{p})$ by

$$\frac{\|g\mathbf{1}_{B(\mathfrak{p})}\|_{p'}}{\mu(B(\mathfrak{p}))} \Big\| \sum_{\mathfrak{p}\in\mathfrak{A}(\mathfrak{p})} (1 + d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{p}'))^{-1/(2a^2+a^3)} \mathbf{1}_{E(\mathfrak{p})} \mathbf{1}_G \Big\|_p.$$

Then we apply Lemma 1.6.1.6 to estimate this by

$$\leq 2^{5a}\frac{\|g\mathbf{1}_{B(\mathfrak{p})}\|_{p'}}{\mu(B(\mathfrak{p}))}\,\mathrm{dens}_1(\mathfrak{A})^{\frac{1}{p}}\mu(B(\mathfrak{p}))^{\frac{1}{p}}.$$

Let $\mathcal{B}'$ be the collection of all balls $B(\mathfrak{p})$ with $\mathfrak{p}\in\mathfrak{A}$. Then for each $\mathfrak{p}\in\mathfrak{A}$ and $x\in B(\mathfrak{p})$ we have by definition `def-hlm` of $M_{\mathcal{B}',p'}$

$$\|g\mathbf{1}_{B(\mathfrak{p})}\|_{p'}\leq \mu(B(\mathfrak{p}))^{\frac{1}{p'}}M_{\mathcal{B}',p'}g(x).$$

Hence we can estimate `eqttt5` by

$$\leq 2^{5a}(M_{\mathcal{B}',p'}g(x))\,\mathrm{dens}_1(\mathfrak{A})^{\frac{1}{p}}.$$

With this estimate of $h(\mathfrak{p})$, using $E(\mathfrak{p})\subset B(\mathfrak{p})$ by construction of $B(\mathfrak{p})$, we estimate `eqtts3` by

$$\leq 2^{232a^3+11a+1}\mathrm{dens}_1(\mathfrak{A})^{\frac{1}{p}}\sum_{\mathfrak{p}\in\mathfrak{A}}\int_{E(\mathfrak{p})}|g|(y)M_{\mathcal{B}',p'}g(y)\,dy.$$

Using Lemma 1.6.1.1, the last display is observed to be

$$\leq 2^{232a^3+11a+1}\mathrm{dens}_1(\mathfrak{A})^{\frac{1}{p}}\int|g|(y)(M_{\mathcal{B}',p'}g)(y)\,dy.$$

Applying Cauchy-Schwarz and using Theorem 1.2.6 and $1<p'<\frac{3}{2}$ estimates the last display by

$$\leq 2^{232a^3+11a+1}\,\mathrm{dens}_1(\mathfrak{A})^{\frac{1}{p}}\|g\|_2\|M_{\mathcal{B}',p'}g\|_2$$

$$\leq 2^{232a^3+12a+3}\,\mathrm{dens}_1(\mathfrak{A})^{\frac{1}{p}}\|g\|_2^2.$$

Now Lemma 1.6.1.4 follows by applying Cauchy-Schwarz on the left-hand side and using $a\geq 4$.

The following basic TT^* estimate will be proved in section 1.6 (p. 65).

Lemma 1.6.1.5

Let $\mathfrak{p},\mathfrak{p}'\in\mathfrak{P}$ with $\mathrm{s}(\mathfrak{p}')\leq\mathrm{s}(\mathfrak{p})$. Then

$$\left|\int T_{\mathfrak{p}'}^*g\overline{T_{\mathfrak{p}}^*g}\right|$$

$$\leq 2^{232a^3}\frac{(1+d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'),\mathcal{Q}(\mathfrak{p}))^{-1/(2a^2+a^3)}}{\mu(\mathcal{I}(\mathfrak{p}))}\int_{E(\mathfrak{p}')}|g|\int_{E(\mathfrak{p})}|g|.$$

Moreover, the term `eq-basic-TT*-est` vanishes unless

$$\mathcal{I}(\mathfrak{p}')\subset B(\mathrm{c}(\mathfrak{p}),14D^{\mathrm{s}(\mathfrak{p})}).$$

The following lemma will be proved in section 1.6 (p. 69).
Lemma 1.6.1.6

Set $p := 4a^4$. For every $\vartheta \in \Theta$ and every antichain $\mathfrak{A}$ we have

$$\Big\| \sum_{\mathfrak{p} \in \mathfrak{A}} (1 + d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \vartheta))^{-1/(2a^2+a^3)} \mathbf{1}_{E(\mathfrak{p})} \mathbf{1}_G \Big\|_p$$

$$\le 2^{5a} \operatorname{dens}_1(\mathfrak{A})^{\frac{1}{p}} \mu \left(\cup_{\mathfrak{p} \in \mathfrak{A}} I_{\mathfrak{p}} \right)^{\frac{1}{p}} .$$

From these lemmas it is easy to prove Theorem 1.2.3.
Proof for Theorem 1.2.3

Proof of Theorem 1.2.3. We have

$$\left(\frac{1}{\tilde{q}} - \frac{1}{2} \right) (2 - q) = \frac{1}{q} - \frac{1}{2}.$$

Multiplying the $(2-q)$-th power of `eqttt9` and the $(q-1)$-th power of `eqttt3` and estimating gives after simplification of some factors

$$\Big| \int \overline{g(x)} \sum_{\mathfrak{p} \in \mathfrak{A}} T_{\mathfrak{p}} f(x) \, d\mu(x) \Big|$$

$$\le 2^{117a^3} (q-1)^{-1} \operatorname{dens}_1(\mathfrak{A})^{\frac{q-1}{2p}} \operatorname{dens}_2(\mathfrak{A})^{\frac{1}{q} - \frac{1}{2}} \|f\|_2 \|g\|_2.$$

With the definition of p, this implies Theorem 1.2.3.

Proof of the Tile Correlation Lemma

The next lemma prepares an application of Theorem 1.2.5.
Lemma 1.6.2.1

Let $-S \le s_1 \le s_2 \le S$ and let $x_1, x_2 \in X$. Define

$$\varphi(y) := \overline{K_{s_1}(x_1, y)} K_{s_2}(x_2, y).$$

If $\varphi(y) \neq 0$, then

$$y \in B(x_1, D^{s_1}).$$

Moreover, we have with $\tau = 1/a$

$$\|\varphi\|_{C^\tau(B(x_1, 2D^{s_1}))} \le \frac{2^{231a^3}}{\mu(B(x_1, D^{s_1}))\mu(B(x_2, D^{s_2}))}.$$

Proof for Lemma 1.6.2.1

Proof. If $\varphi(y)$ is not zero, then $K_{s_1}(x_1, y)$ is not zero and thus `supp-Ks` gives `eqt10`.We next have for y with `eq-Ks-size`

$$|\varphi(y)| \le \frac{2^{204a^3}}{\mu(B(x_1, D^{s_1}))\mu(B(x_2, D^{s_2}))}$$

and for $y' \neq y$ additionally with `eq-Ks-smooth`

$$|\varphi(y) - \varphi(y')|$$
$$\le |K_{s_1}(x_1, y) - K_{s_1}(x_1, y')||K_{s_2}(x_2, y)|$$
$$+|K_{s_1}(x_1, y')||K_{s_2}(x_2, y) - K_{s_2}(x_2, y')|$$
$$\le \frac{2^{229a^3}}{\mu(B(x_1, D^{s_1}))\mu(B(x_2, D^{s_2}))}\left(\left(\frac{\rho(y, y')}{D^{s_1}}\right)^{1/a} + \left(\frac{\rho(y, y')}{D^{s_2}}\right)^{1/a}\right)$$
$$\le \frac{2^{230a^3}}{\mu(B(x_1, D^{s_1}))\mu(B(x_2, D^{s_2}))}\left(\frac{\rho(y, y')}{D^{s_1}}\right)^{1/a}.$$

Adding the estimates `suppart` and `holderpart` gives `eqt11`. This proves the lemma.

The following auxiliary statement about the support of $T_{\mathfrak{p}}^* g$ will be used repeatedly.

Lemma 1.6.2.2

For each $\mathfrak{p} \in \mathfrak{P}$, and each $y \in X$, we have that

$$T_{\mathfrak{p}}^* g(y) \neq 0$$

implies

$$y \in B(\mathtt{c}(\mathfrak{p}), 5D^{\mathtt{s}(\mathfrak{p})}).$$

Proof for Lemma 1.6.2.2

Proof. Fix $\mathfrak{p}$ and y with `tstargnot0`. Then there exists $x \in E(\mathfrak{p})$ with

$$\overline{K_{\mathtt{s}(\mathfrak{p})}(x, y)} e(-Q(x)(y) + Q(x)(x)) g(x) \neq 0.$$

As $E(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{p})$ and by the squeezing property `eq-vol-sp-cube`, we have

$$\rho(x, \mathtt{c}(\mathfrak{p})) < 4D^{\mathtt{s}(\mathfrak{p})}.$$

As $K_{\mathtt{s}(\mathfrak{p})}(x, y) \neq 0$, we have by `supp-Ks` that

$$\rho(x, y) \le \frac{1}{2} D^{\mathtt{s}(\mathfrak{p})}.$$

Now `ynotfar` follows by the triangle inequality.

The next lemma is a geometric estimate for two tiles.

Lemma 1.6.2.3

Let $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathfrak{P}$ with $B(\mathsf{c}(\mathfrak{p}_1), 5D^{\mathsf{s}(\mathfrak{p}_1)}) \cap B(\mathsf{c}(\mathfrak{p}_2), 5D^{\mathsf{s}(\mathfrak{p}_2)}) \neq \emptyset$ and $\mathsf{s}(\mathfrak{p}_1) \leq \mathsf{s}(\mathfrak{p}_2)$. For each $x_1 \in E(\mathfrak{p}_1)$ and $x_2 \in E(\mathfrak{p}_2)$ we have

$$1 + d_{\mathfrak{p}_1}(\mathcal{Q}(\mathfrak{p}_1), \mathcal{Q}(\mathfrak{p}_2)) \leq 2^{8a}(1 + d_{B(x_1, D^{\mathsf{s}(\mathfrak{p}_1)})}(Q(x_1), Q(x_2))).$$

Proof for Lemma 1.6.2.3

Proof. Let $i \in \{1, 2\}$. By definition `defineep` of E, we have $Q(x_i) \in \Omega(\mathfrak{p}_i)$. With `eq-freq-comp-ball` we then conclude

$$d_{\mathfrak{p}_i}(Q(x_i), \mathcal{Q}(\mathfrak{p}_i)) < 1.$$

We have by the triangle inequality and `eq-vol-sp-cube` that $\mathcal{I}(\mathfrak{p}_1) \subset B(\mathsf{c}(\mathfrak{p}_2), 14D^{\mathsf{s}(\mathfrak{p}_2)})$. Thus, using again `eq-vol-sp-cube` and the doubling property `firstdb`

$$d_{\mathfrak{p}_1}(Q(x_2), \mathcal{Q}(\mathfrak{p}_2)) \leq 2^{6a} d_{\mathfrak{p}_2}(Q(x_2), \mathcal{Q}(\mathfrak{p}_2)) \leq 2^{6a}.$$

By the triangle inequality, we obtain from `dponetwo` and `tgeo0.5`

$$1 + d_{\mathfrak{p}_1}(\mathcal{Q}(\mathfrak{p}_1), \mathcal{Q}(\mathfrak{p}_2)) \leq 2 + 2^{6a} + d_{\mathfrak{p}_1}(Q(x_1), Q(x_2)).$$

As $x_1 \in \mathcal{I}(\mathfrak{p}_1)$ by definition `defineep` of E, we have by the squeezing property `eq-vol-sp-cube`

$$d(x_1, \mathsf{c}(\mathfrak{p}_1)) \leq 4D^{\mathsf{s}(\mathfrak{p}_1)}$$

and thus by `eq-vol-sp-cube` again and the triangle inequality

$$\mathcal{I}(\mathfrak{p}_1) \subset B(x_1, 8D^{\mathsf{s}(\mathfrak{p}_1)}).$$

We thus estimate the right-hand side of `tgeo1` with monotonicity `monotonedb` of the metrics d_B by

$$\leq 2 + 2^{6a} + d_{B(x_1, 8D^{\mathsf{s}(\mathfrak{p}_1)})}(Q(x_1), Q(x_2)).$$

This is further estimated by applying the doubling property `firstdb` three times by

$$\leq 2 + 2^{6a} + 2^{3a} d_{B_1(x_1, D^{\mathsf{s}(\mathfrak{p}_1)})}(Q(x_1), Q(x_2)).$$

Now `tgeo` follows with $a \geq 4$.

We now prove Lemma 1.6.1.5.

Proof for Lemma 1.6.1.5

Proof of Lemma 1.6.1.5. We begin with `eq-basic-TT*-est`. By Lemma 1.6.2.2, the left-hand side of `eq-basic-TT*-est` vanishes if $B(\mathrm{c}(\mathfrak{p}'), 5D^{\mathrm{s}(\mathfrak{p}')}) \cap B(\mathrm{c}(\mathfrak{p}), 5D^{\mathrm{s}(\mathfrak{p})}) = \emptyset$. Thus we can assume for the remainder of the proof that

$$B(\mathrm{c}(\mathfrak{p}'), 5D^{\mathrm{s}(\mathfrak{p}')}) \cap B(\mathrm{c}(\mathfrak{p}), 5D^{\mathrm{s}(\mathfrak{p})}) \neq \emptyset.$$

We expand the left-hand side of `eq-basic-TT*-est` as

$$\Bigg| \int \int_{E(\mathfrak{p}')} \overline{K_{\mathrm{s}(\mathfrak{p}')}(x_1, y)} e(-Q(x_1)(y) + Q(x_1)(x_1)) g(x_1)\, d\mu(x_1)$$

$$\times \int_{E(\mathfrak{p})} K_{\mathrm{s}(\mathfrak{p})}(x_2, y) e(Q(x_2)(y) - Q(x_2)(x_2)) \overline{g(x_2)}\, d\mu(x_2)\, d\mu(y) \Bigg|.$$

By Fubini and the triangle inequality and the fact $|e(Q(x_i)(x_i))| = 1$ for $i = 1, 2$, we can estimate `tstartstar` and `tstartstar'` from above by

$$\int_{E(\mathfrak{p}')} \int_{E(\mathfrak{p})} \mathbf{I}(x_1, x_2)\, d\mu(x_1) d\mu(x_2),$$

with

$$\mathbf{I}(x_1, x_2) := \left| \int e(-Q(x_1)(y) + Q(x_2)(y)) \varphi_{x_1, x_2}(y) d\mu(y)\, g(x_1) g(x_2) \right|.$$

We estimate for fixed $x_1 \in E(\mathfrak{p}')$ and $x_2 \in E(\mathfrak{p})$ the inner integral of `eqa1` with Theorem 1.2.5. The function $\varphi := \varphi_{x_1,x_2}$ satisfies the assumptions of Theorem 1.2.5 with $z = x_1$ and $R = D^{s_1}$ by Lemma 1.6.2.1. We obtain with $B' := B(x_1, D^{\mathrm{s}(\mathfrak{p}')})$

$$\mathbf{I}(x_1, x_2) \leq 2^{8a} \mu(B') \|\varphi\|_{C^\tau(B')} (1 + d_{B'}(Q(x_1), Q(x_2)))^{-1/(2a^2+a^3)} |g(x_1) g(x_2)|$$

$$\leq \frac{2^{231a^3+8a}}{\mu(B(x_2, D^{\mathrm{s}(\mathfrak{p})}))} (1 + d_{B'}(Q(x_1), Q(x_2)))^{-1/(2a^2+a^3)}.$$

Using `intersec5B`, Lemma 1.6.2.3 and $a \geq 1$ estimates `eqa1.5` by

$$\leq \frac{2^{231a^3+8a+1}}{\mu(B(x_2, D^{\mathrm{s}(\mathfrak{p})}))} (1 + d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \mathcal{Q}(\mathfrak{p})))^{-1/(2a^2+a^3)} |g(x_1) g(x_2)|.$$

As $x_2 \in \mathcal{I}(\mathfrak{p})$ by definition `defineep` of E, we have by `eq-vol-sp-cube`

$$\rho(x_2, \mathrm{c}(\mathfrak{p})) < 4D^{\mathrm{s}(\mathfrak{p})}$$

and thus by `eq-vol-sp-cube` again and the triangle inequality

$$\mathcal{I}(\mathfrak{p}) \subset B(x_2, 8D^{\mathrm{s}(\mathfrak{p})}).$$

Using three iterations of the doubling property `doublingx` give

$$\mu(\mathcal{I}(\mathfrak{p})) \le 2^{3a}\mu(B(x_2, D^{\mathrm{s}(\mathfrak{p})})).$$

With $a \ge 4$ and `eqa2` we conclude `eq-basic-TT*-est`.Now assume the left-hand side of `eq-basic-TT*-est` is not zero. There is a $y \in X$ with

$$T^*_{\mathfrak{p}'}g(y)\overline{T^*_{\mathfrak{p}}g(y)} \neq 0$$

By the triangle inequality and Lemma 1.6.2.2, we conclude

$$\rho(\mathtt{c}(\mathfrak{p}), \mathtt{c}(\mathfrak{p}')) \le \rho(\mathtt{c}(\mathfrak{p}), y) + \rho(\mathtt{c}(\mathfrak{p}'), y) \le 5D^{\mathrm{s}(\mathfrak{p})} + 5D^{\mathrm{s}(\mathfrak{p}')} \le 10D^{\mathrm{s}(\mathfrak{p})}.$$

By the squeezing property `eq-vol-sp-cube` and the triangle inequality, we conclude

$$\mathcal{I}(\mathfrak{p}') \subset B(\mathtt{c}(\mathfrak{p}), 14D^{\mathrm{s}(\mathfrak{p})}).$$

This completes the proof of Lemma 1.6.1.5.

Proof of the Antichain Tile Count Lemma

Lemma 1.6.3.1

Let $\vartheta \in \Theta$ and $N \ge 0$ be an integer. Let $\mathfrak{p}, \mathfrak{p}' \in \mathfrak{P}$ with

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \vartheta) \le 2^N$$

$$d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \vartheta) \le 2^N.$$

Assume $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{p}')$ and $\mathrm{s}(\mathfrak{p}) < \mathrm{s}(\mathfrak{p}')$. Then

$$2^{N+2}\mathfrak{p} \lesssim 2^{N+2}\mathfrak{p}'.$$

Proof for Lemma 1.6.3.1

Proof. By Lemma 1.2.1.2, we have

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}'), \vartheta) \le d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \vartheta) \le 2^N.$$

Together with `eqassumedismfa` and the triangle inequality, we obtain

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}'), \mathcal{Q}(\mathfrak{p})) \le 2^{N+1}.$$

Now assume

$$\vartheta' \in B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), 2^{N+2}).$$

By the doubling property `firstdb`, applied five times, we have

$$d_{B(\mathtt{c}(\mathfrak{p}'), 8D^{\mathrm{s}(\mathfrak{p}')})}(\mathcal{Q}(\mathfrak{p}'), \vartheta') < 2^{5a+N+2}.$$

We have by the squeezing property `eq-vol-sp-cube`

$$\mathsf{c}(\mathfrak{p}) \in B(\mathsf{c}(\mathfrak{p}'), 4D^{\mathsf{s}(\mathfrak{p}')}).$$

Hence by the triangle inequality

$$B(\mathsf{c}(\mathfrak{p}), 4D^{\mathsf{s}(\mathfrak{p}')}) \subseteq B(\mathsf{c}(\mathfrak{p}'), 8D^{\mathsf{s}(\mathfrak{p}')}).$$

Together with `ageo1` and monotonicity `monotonedb` of d

$$d_{B(\mathsf{c}(\mathfrak{p}), 4D^{\mathsf{s}(\mathfrak{p}')})}(\mathcal{Q}(\mathfrak{p}'), \vartheta') < 2^{5a+N+2}.$$

Using the doubling property `seconddb` $5a+2$ times gives

$$d_{B(\mathsf{c}(\mathfrak{p}), 2^{2-5a^2-2a}D^{\mathsf{s}(\mathfrak{p}')})}(\mathcal{Q}(\mathfrak{p}'), \vartheta') < 2^{N}.$$

Using $\mathsf{s}(\mathfrak{p}) < \mathsf{s}(\mathfrak{p}')$ and $D = 2^{100a^2}$ and $a \geq 4$ gives

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}'), \vartheta') < 2^{N}.$$

With the triangle inequality and `eqdistqpqp`,

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \vartheta') < 2^{N+2}.$$

This shows

$$B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), 2^{N+2}) \subset B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 2^{N+2}).$$

This implies `lp'lp''` and completes the proof of the lemma.

For $\vartheta \in \Theta$ and $N \geq 0$ define

$$\mathfrak{A}_{\vartheta,N} := \{\mathfrak{p} \in \mathfrak{A} : 2^N \leq 1 + d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \vartheta) < 2^{N+1}\}.$$

Lemma 1.6.3.2

Let $\vartheta \in \Theta$, $N \geq 0$ and $L \in \mathcal{D}$. Then

$$\sum_{\mathfrak{p} \in \mathfrak{A}_{\vartheta,N} : \mathcal{I}(\mathfrak{p}) = L} \mu(E(\mathfrak{p}) \cap G) \leq 2^{a(N+5)} \operatorname{dens}_1(\mathfrak{A}) \mu(L).$$

Proof for Lemma 1.6.3.2

Proof. Let ϑ, N, L be given and set

$$\mathfrak{A}' := \{\mathfrak{p} \in \mathfrak{A}_{\vartheta,N} : \mathcal{I}(\mathfrak{p}) = L\}.$$

Let $\mathfrak{p} \in \mathfrak{A}'$. We have by definition `definedens1` using $\lambda = 2$ and the squeezing property `eq-freq-comp-ball`

$$\mu(E(\mathfrak{p}) \cap G) \leq \mu(E_2(2, \mathfrak{p})) \leq 2^a \operatorname{dens}_1(\mathfrak{A}') \mu(L).$$

By the covering property `thirddb`, applied $N+4$ times, there is a collection Θ' of at most $2^{a(N+4)}$ elements such that

$$B_{\mathfrak{p}}(\vartheta, 2^{N+1}) \subset \bigcup_{\vartheta' \in \Theta'} B_{\mathfrak{p}}(\vartheta', 0.2).$$

As each $\mathcal{Q}(\mathfrak{p}')$ with $\mathfrak{p}' \in \mathfrak{A}'$ is contained in the left-hand side of `eqanti-4` by definition, it is in at least one $B_{\mathfrak{p}}(\vartheta', 0.2)$ with $\vartheta' \in \Theta'$.For two different $\mathfrak{p}', \mathfrak{p}'' \in \mathfrak{A}'$, we have by `eq-dis-freq-cover` that $\Omega(\mathfrak{p}')$ and $\Omega(\mathfrak{p}'')$ are disjoint and thus by the squeezing property `eq-freq-comp-ball` we have for every $\vartheta' \in \Theta'$

$$\vartheta' \notin B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}'), 0.2) \cap B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}''), 0.2).$$

Hence at most one of $\mathcal{Q}(\mathfrak{p}')$ and $\mathcal{Q}(\mathfrak{p}'')$ is in $B_{\mathfrak{p}}(\vartheta', 0.2)$. It follows that there are at most $2^{a(N+4)}$ elements in $\mathfrak{A}'$. Adding `eqanti-3` over $\mathfrak{A}'$ proves `eqanti-1`.

Lemma 1.6.3.3

Let $\vartheta \in \Theta$ and N be an integer. Let $\mathfrak{p}_\vartheta$ be a tile with $\vartheta \in B_{\mathfrak{p}_\vartheta}(\mathcal{Q}(\mathfrak{p}_\vartheta), 2^{N+1})$. Then we have

$$\sum_{\mathfrak{p} \in \mathfrak{A}_{\vartheta,N} : \mathrm{s}(\mathfrak{p}_\vartheta) < \mathrm{s}(\mathfrak{p})} \mu(E(\mathfrak{p}) \cap G \cap \mathcal{I}(\mathfrak{p}_\vartheta)) \le \mu(E_2(2^{N+3}, \mathfrak{p}_\vartheta)).$$

Proof for Lemma 1.6.3.3

Proof. Let $\mathfrak{p}$ be any tile in $\mathfrak{A}_{\vartheta,N}$ with $\mathrm{s}(\mathfrak{p}_\vartheta) < \mathrm{s}(\mathfrak{p})$. By definition of E, the tile contributes zero to the sum on the left-hand side of `eqanti-0.5` unless $\mathcal{I}(\mathfrak{p}) \cap \mathcal{I}(\mathfrak{p}_\vartheta) \neq \emptyset$, which we may assume. With $\mathrm{s}(\mathfrak{p}_\vartheta) < \mathrm{s}(\mathfrak{p})$ and the dyadic property `dyadicproperty` we conclude $\mathcal{I}(\mathfrak{p}_\vartheta) \subset \mathcal{I}(\mathfrak{p})$. We conclude from $\mathfrak{p} \in \mathfrak{A}_{\vartheta,N}$ that

$$\vartheta \in B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 2^{N+1}).$$

With Lemma 1.6.3.1 and the assumption on $\mathfrak{p}_\vartheta$, we conclude

$$2^{N+3}\mathfrak{p}_\vartheta \lesssim 2^{N+3}\mathfrak{p}.$$

By definition `definee2` of E_2, we conclude

$$E(\mathfrak{p}) \cap G \cap \mathcal{I}(\mathfrak{p}_\vartheta) \subset E_2(2^{N+3}, \mathfrak{p}_\vartheta).$$

Using disjointedness of the various $E(\mathfrak{p})$ with $\mathfrak{p} \in \mathfrak{A}$ by Lemma 1.6.1.1, we obtain `eqanti-0.5`. This proves the lemma.

Lemma 1.6.3.4

Let $\vartheta \in Q(X)$ and let $N \geq 0$ be an integer. Then we have

$$\sum_{\mathfrak{p}\in\mathfrak{A}_{\vartheta,N}} \mu(E(\mathfrak{p})\cap G) \leq 2^{101a^3+Na}\,\mathrm{dens}_1(\mathfrak{A})\mu\left(\cup_{\mathfrak{p}\in\mathfrak{A}} I_{\mathfrak{p}}\right).$$

Proof for Lemma 1.6.3.4

Proof. Fix ϑ and N. Let $\mathfrak{A}'$ be the set of $\mathfrak{p} \in \mathfrak{A}_{\vartheta,N}$ such that $\mathcal{I}(\mathfrak{p})\cap G$ is not empty and $s(\mathfrak{p}) > -S$. Let $\mathfrak{A}_{-S}$ be the set of $\mathfrak{p} \in \mathfrak{A}_{\vartheta,N}$ such that $\mathcal{I}(\mathfrak{p})\cap G$ is not empty and $s(\mathfrak{p}) = -S$. Then we have

$$\sum_{\mathfrak{p}\in\mathfrak{A}_{\vartheta,N}} \mu(E(\mathfrak{p})\cap G) = \sum_{\mathfrak{p}\in\mathfrak{A}'} \mu(E(\mathfrak{p})\cap G) + \sum_{\mathfrak{p}\in\mathfrak{A}_{-S}} \mu(E(\mathfrak{p})\cap G).$$

We start by estimating the contribution of $\mathfrak{A}_{-S}$. Let $\mathcal{L}_{-S}$ be the collection of dyadic cubes $\mathcal{I}(\mathfrak{p})$ with $\mathfrak{p} \in \mathfrak{A}_{-S}$. They all have scale $-S$, by definition of $\mathcal{L}_{-S}$, and hence they are pairwise disjoint by the dyadic property `dyadicproperty`. We write

$$\sum_{\mathfrak{p}\in\mathfrak{A}_{-S}} \mu(E(\mathfrak{p})\cap G) = \sum_{L\in\mathcal{L}_{-S}} \sum_{\mathfrak{p}\in\mathfrak{A}_{-S},\mathcal{I}(\mathfrak{p})=L} \mu(E(\mathfrak{p})\cap G),$$

and using Lemma 1.6.3.2, we estimate

$$\leq 2^{a(N+5)}\,\mathrm{dens}_1(\mathfrak{A}) \sum_{L\in\mathcal{L}_{-S}} \mu(L) \leq 2^{a(N+5)}\,\mathrm{dens}_1(\mathfrak{A})\mu\left(\cup_{\mathfrak{p}\in\mathfrak{A}} I_{\mathfrak{p}}\right).$$

We turn to $\mathfrak{A}'$. Let $\mathcal{L}$ be the collection of dyadic cubes $I \in \mathcal{D}$ such that $I \leq \mathcal{I}(\mathfrak{p})$ for some $\mathfrak{p} \in \mathfrak{A}'$ and such that $\mathcal{I}(\mathfrak{p}) \not\subset I$ for all $\mathfrak{p} \in \mathfrak{A}'$. By `coverdyadic`, for each $\mathfrak{p} \in \mathfrak{A}'$ and each $x \in \mathcal{I}(\mathfrak{p})\cap G$, there is $I \in \mathcal{D}$ with $s(I) = -S$ and $x \in I$. By `dyadicproperty`, we have $I \subset \mathcal{I}(\mathfrak{p})$. Since for all $\mathfrak{p}' \in \mathfrak{A}'$ we have $s(\mathfrak{p}') > -S$, we have $I \in \mathcal{L}$. Hence

$$\mathcal{I}(\mathfrak{p}) \subset \bigcup\{I \in \mathcal{D} : s(I) = -S, I \subset \mathcal{I}(\mathfrak{p})\} \subset \bigcup \mathcal{L}.$$

As each $I \in \mathcal{L}$ satisfies $I \subset \mathcal{I}(\mathfrak{p})$ for some $\mathfrak{p}$ in $\mathfrak{A}'$, we conclude

$$\bigcup \mathcal{L} = \bigcup_{\mathfrak{p}\in\mathfrak{A}'} \mathcal{I}(\mathfrak{p}).$$

Let $\mathcal{L}^*$ be the set of maximal elements in $\mathcal{L}$ with respect to set inclusion. By `dyadicproperty`, the elements in $\mathcal{L}^*$ are pairwise disjoint and we have

$$\bigcup \mathcal{L}^* = \bigcup_{\mathfrak{p}\in\mathfrak{A}'} \mathcal{I}(\mathfrak{p}).$$

Using the partition eqdecAprime into elements of $\mathcal{L}$ in eqanti0, it suffices to show for each $L \in \mathcal{L}^*$

$$\sum_{\mathfrak{p} \in \mathfrak{A}'} \mu(E(\mathfrak{p}) \cap G \cap L) \leq 2^{101a^3+aN} \operatorname{dens}_1(\mathfrak{A})\mu(L).$$

Fix $L \in \mathcal{L}^*$. By definition of $\mathcal{L}^*$, there exists an element $\mathfrak{p}' \in \mathfrak{A}'$ such that $L \subset \mathcal{I}(\mathfrak{p}')$. Pick such an element $\mathfrak{p}'$ in $\mathfrak{A}$ with minimal $\mathrm{s}(\mathfrak{p}')$. As $\mathcal{I}(\mathfrak{p}') \not\subset L$ by definition of L, we have with dyadicproperty that $s(L) < \mathrm{s}(\mathfrak{p}')$. In particular $s(L) < S$, thus $L \neq I_0$ and hence by subsetmaxcube there exists a cube $J \in \mathcal{D}$ with $L \subsetneq J$. By coverdyadic, there is an $L' \in \mathcal{D}$ with $s(L') = s(L)+1$ and $L \leq L'$. By dyadicproperty, we have $L \subset L'$.We split the left-hand side of eqanti0 as

$$\sum_{\mathfrak{p} \in \mathfrak{A}' : \mathcal{I}(\mathfrak{p}) = L'} \mu(E(\mathfrak{p}) \cap G \cap L)$$

$$+ \sum_{\mathfrak{p} \in \mathfrak{A}' : \mathcal{I}(\mathfrak{p}) \neq L'} \mu(E(\mathfrak{p}) \cap G \cap L).$$

We first estimate eqanti1 with Lemma 1.6.3.2 by

$$\leq \sum_{\mathfrak{p} \in \mathfrak{A}' : \mathcal{I}(\mathfrak{p}) = L'} \mu(E(\mathfrak{p}) \cap G \cap L') \leq 2^{a(N+5)} \operatorname{dens}_1(\mathfrak{A})\mu(L').$$

We turn to eqanti2. Consider the element $\mathfrak{p}' \in \mathfrak{A}'$ as above with $L \subset \mathcal{I}(\mathfrak{p}')$ and $s(L) < \mathrm{s}(\mathfrak{p}')$. As $L \subset L'$ and $s(L') = s(L) + 1$, we conclude with the dyadic property that $L' \subset \mathcal{I}(\mathfrak{p}')$. By maximality of L, we have $L' \notin \mathcal{L}$. This together with the existence of the given $\mathfrak{p}' \in \mathfrak{A}$ with $L' \subset \mathcal{I}(\mathfrak{p}')$ shows by definition of $\mathcal{L}$ that there exists $\mathfrak{p}'' \in \mathfrak{A}'$ with $\mathcal{I}(\mathfrak{p}'') \subset L'$.If $\mathcal{I}(\mathfrak{p}'') = L'$, then we set $\mathfrak{p}_\vartheta = \mathfrak{p}''$ and note that as $\mathfrak{p}'' \in \mathfrak{A}_{\vartheta,N}$

$$\vartheta \in B(\mathcal{Q}(\mathfrak{p}''), 2^{N+1}).$$

If $\mathcal{I}(\mathfrak{p}'') \neq L'$, then it follows that $s(\mathfrak{p}'') < s(L')$. By the covering property eq-dis-freq-cover, there exists a unique $\mathfrak{p}_\vartheta$ with

$$\mathcal{I}(\mathfrak{p}_\vartheta) = L'$$

such that $\vartheta \in \Omega(\mathfrak{p}_\vartheta)$. We take this as the definition of $\mathfrak{p}_\vartheta$ in this case. Note that

$$\vartheta \in B(\mathcal{Q}(\mathfrak{p}_\vartheta), 1)$$

so by Lemma 1.6.3.1, we conclude

$$2^{N+3}\mathfrak{p}'' \lesssim 2^{N+3}\mathfrak{p}_\vartheta.$$

This clearly also holds in the case $\mathcal{I}(\mathfrak{p}'') = L'$, since then $\mathfrak{p}'' = \mathfrak{p}_\vartheta$. Furthermore, in both cases it also holds that

$$\vartheta \in B_{\mathfrak{p}_\vartheta}(\mathcal{Q}(\mathfrak{p}_\vartheta), 2^{N+1}).$$

As $\mathfrak{p}'' \in \mathfrak{A}'$, we have by definition `definedens1` of dens_1 that

$$\mu(E_2(2^{N+3}, \mathfrak{p}_\vartheta)) \leq 2^{Na+3a} \,\mathrm{dens}_1(\mathfrak{A})\mu(L').$$

Now let $\mathfrak{p}$ be any tile in the summation set in `eqanti2`, that is, $\mathfrak{p} \in \mathfrak{A}'$ and $\mathcal{I}(\mathfrak{p}) \neq L'$. Then $\mathcal{I}(\mathfrak{p}) \cap L \neq \emptyset$. It follows by the dyadic property `dyadicproperty` and the definition of L that $L \subset \mathcal{I}(\mathfrak{p})$ and $L \neq \mathcal{I}(\mathfrak{p})$. By `dyadicproperty`, we have $s(L) < \mathrm{s}(\mathfrak{p})$ and thus $s(L') \leq \mathrm{s}(\mathfrak{p})$. By `dyadicproperty` again, we have $L' \subset \mathcal{I}(\mathfrak{p})$. As $L' \neq \mathcal{I}(\mathfrak{p})$, we conclude $s(L) < \mathrm{s}(\mathfrak{p})$. By Lemma 1.6.3.3, we can thus estimate `eqanti2` by

$$\sum_{\mathfrak{p} \in \mathfrak{A}' : \mathcal{I}(\mathfrak{p}) \neq L'} \mu(E(\mathfrak{p}) \cap G \cap L') \leq \mu(E_2(2^{N+3}, \mathfrak{p}_\vartheta)).$$

With the decomposition in `eqanti1` and `eqanti2` and the estimates `equanti1.5`, `eqanti-0.5`, `pmfadens` we obtain the estimate

$$\sum_{\mathfrak{p} \in \mathfrak{A}'} \mu(E(\mathfrak{p}) \cap G \cap L) \leq (2^{a(N+5)} + 2^{Na+3a}) \,\mathrm{dens}_1(\mathfrak{A})\mu(L').$$

Using $s(L') = s(L) + 1$ and $D = 2^{100a^2}$ and the squeezing property `eq-vol-sp-cube` and the doubling property `doublingx` $100a^2 + 4$ times, we obtain

$$\mu(L') \leq 2^{100a^3+4a}\mu(L).$$

Inserting in `eqanti3.14`, adding the estimate `eq_minus_S` and using $a \geq 4$ gives `eqanti0`. This completes the proof of the lemma.

We turn to the proof of Lemma 1.6.1.6.

Proof for Lemma 1.6.1.6

Proof of Lemma 1.6.1.6. Using that $\mathfrak{A}$ is the disjoint union of the $\mathfrak{A}_{\vartheta,N}$ with $N \geq 0$, we estimate the left-hand side `eq-antichain-Lp` with the triangle inequality by

$$\leq \sum_{N \geq 0} \left\| \sum_{\mathfrak{p} \in \mathfrak{A}_{\vartheta,N}} 2^{-N/(2a^2+a^3)} \mathbf{1}_{E(\mathfrak{p})} \mathbf{1}_G \right\|_p .$$

We consider each individual term in this sum and estimate its p-th power. Using that for each $x \in X$ by Lemma 1.6.1.1 there is at most one $\mathfrak{p} \in \mathfrak{A}$ with $x \in E(\mathfrak{p})$, we have

$$\left\| \sum_{\mathfrak{p} \in \mathfrak{A}_{\vartheta,N}} 2^{-N/(2a^2+a^3)} \mathbf{1}_{E(\mathfrak{p})} \mathbf{1}_G \right\|_p^p$$

$$= \int \Big(\sum_{\mathfrak{p} \in \mathfrak{A}_{\vartheta,N}} 2^{-N/(2a^2+a^3)} \mathbf{1}_{E(\mathfrak{p}) \cap G}(x) \Big)^p \, d\mu(x)$$

$$= \int \sum_{\mathfrak{p} \in \mathfrak{A}_{\vartheta,N}} 2^{-pN/(2a^2+a^3)} \mathbf{1}_{E(\mathfrak{p}) \cap G}(x) \, d\mu(x)$$

$$= 2^{-pN/(2a^2+a^3)} \sum_{\mathfrak{p} \in \mathfrak{A}_{\vartheta,N}} \mu(E(\mathfrak{p}) \cap G).$$

Using Lemma 1.6.3.4, we estimate the last display by

$$\leq 2^{-pN/(2a^2+a^3)+101a^3+Na} \operatorname{dens}_1(\mathfrak{A}) \mu \left(\cup_{\mathfrak{p} \in \mathfrak{A}} \mathcal{I}(\mathfrak{p}) \right).$$

Using that $a \geq 4$ and since $p = 4a^4$, we have

$$pN/(2a^2 + a^3) \geq 4a^4 N/(3a^3) \geq Na + N.$$

Hence we have for `eqanti21` the upper bound

$$\leq 2^{101a^3 - N} \operatorname{dens}_1(\mathfrak{A}) \mu \left(\cup_{\mathfrak{p} \in \mathfrak{A}} \mathcal{I}(\mathfrak{p}) \right).$$

Taking the p-th root and summing over $N \geq 0$ gives for `eqanti23` the upper bound

$$\leq \left(\sum_{N \geq 0} 2^{-N/p} \right) 2^{101a^3/p} \operatorname{dens}_1(\mathfrak{A})^{\frac{1}{p}} \mu \left(\cup_{\mathfrak{p} \in \mathfrak{A}} \mathcal{I}(\mathfrak{p}) \right)^{\frac{1}{p}}$$

$$= \left(1 - 2^{-1/p}\right)^{-1} 2^{101a^3/p} \operatorname{dens}_1(\mathfrak{A})^{\frac{1}{p}} \mu \left(\cup_{\mathfrak{p} \in \mathfrak{A}} \mathcal{I}(\mathfrak{p}) \right)^{\frac{1}{p}}.$$

Using that $p = 4a^4$ and $a \geq 4$, this proves the lemma.

1.7 Proof of the Forest Operator Proposition

The pointwise tree estimate

Fix a forest $(\mathfrak{U}, \mathfrak{T})$. The main result of this subsection is Lemma 1.7.1.3, we begin this section with some definitions necessary to state the lemma.

For $\mathfrak{u} \in \mathfrak{U}$ and $x \in X$, we define

$$\sigma(\mathfrak{u}, x) := \{\mathsf{s}(\mathfrak{p}) : \mathfrak{p} \in \mathfrak{T}(\mathfrak{u}), x \in E(\mathfrak{p})\}.$$

This is a subset of $\mathbb{Z} \cap [-S, S]$, so has a minimum and a maximum. We set

$$\overline{\sigma}(\mathfrak{u}, x) := \max \sigma(\mathfrak{T}(\mathfrak{u}), x)$$

$$\underline{\sigma}(\mathfrak{u}, x) := \min \sigma(\mathfrak{T}(\mathfrak{u}), x).$$

Lemma 1.7.1.1

For each $\mathfrak{u} \in \mathfrak{U}$, we have

$$\sigma(\mathfrak{u}, x) = \mathbb{Z} \cap [\underline{\sigma}(\mathfrak{u}, x), \overline{\sigma}(\mathfrak{u}, x)].$$

Proof for Lemma 1.7.1.1

Proof. Let $s \in \mathbb{Z}$ with $\underline{\sigma}(\mathfrak{u}, x) \leq s \leq \overline{\sigma}(\mathfrak{u}, x)$. By definition of σ, there exists $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ with $\mathrm{s}(\mathfrak{p}) = \underline{\sigma}(\mathfrak{u}, x)$ and $x \in E(\mathfrak{p})$, and there exists $\mathfrak{p}'' \in \mathfrak{T}(\mathfrak{u})$ with $\mathrm{s}(\mathfrak{p}'') = \overline{\sigma}(\mathfrak{u}, x)$ and $x \in E(\mathfrak{p}'') \subset \mathcal{I}(\mathfrak{p}'')$. By property `coverdyadic` of the dyadic grid, there exists a cube $I \in \mathcal{D}$ of scale s with $x \in I$. By property `eq-dis-freq-cover`, there exists a tile $\mathfrak{p}' \in \mathfrak{P}(I)$ with $Q(x) \in \Omega(\mathfrak{p}')$. By the dyadic property `dyadicproperty` we have $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{p}') \subset \mathcal{I}(\mathfrak{p}'')$, and by `eq-freq-dyadic`, we have $\Omega(\mathfrak{p}'') \subset \Omega(\mathfrak{p}') \subset \Omega(\mathfrak{p})$. Thus $\mathfrak{p} \leq \mathfrak{p}' \leq \mathfrak{p}''$, which gives with the convexity property `forest2` of $\mathfrak{T}(\mathfrak{u})$ that $\mathfrak{p}' \in \mathfrak{T}(\mathfrak{u})$, so $s \in \sigma(\mathfrak{u}, x)$.

For a nonempty collection of tiles $\mathfrak{S} \subset \mathfrak{P}$ we define

$$\mathcal{J}_0(\mathfrak{S})$$

to be the collection of all dyadic cubes $J \in \mathcal{D}$ such that $s(J) = -S$ or

$$\mathcal{I}(\mathfrak{p}) \not\subset B(c(J), 100D^{s(J)+1})$$

for all $\mathfrak{p} \in \mathfrak{S}$. We define $\mathcal{J}(\mathfrak{S})$ to be the collection of inclusion maximal cubes in $\mathcal{J}_0(\mathfrak{S})$.

We further define

$$\mathcal{L}_0(\mathfrak{S})$$

to be the collection of dyadic cubes $L \in \mathcal{D}$ such that $s(L) = -S$, or there exists $\mathfrak{p} \in \mathfrak{S}$ with $L \subset \mathcal{I}(\mathfrak{p})$ and there exists no $\mathfrak{p} \in \mathfrak{S}$ with $\mathcal{I}(\mathfrak{p}) \subset L$. We define $\mathcal{L}(\mathfrak{S})$ to be the collection of inclusion maximal cubes in $\mathcal{L}_0(\mathfrak{S})$.

Lemma 1.7.1.2

For each $\mathfrak{S} \subset \mathfrak{P}$, we have

$$\bigcup_{I \in \mathcal{D}} I = \dot{\bigcup_{J \in \mathcal{J}(\mathfrak{S})}} J$$

and

$$\bigcup_{I \in \mathcal{D}} I = \dot{\bigcup_{L \in \mathcal{L}(\mathfrak{S})}} L.$$

Proof for Lemma 1.7.1.2

Proof. Since $\mathcal{J}(\mathfrak{S})$ is the set of inclusion maximal cubes in $\mathcal{J}_0(\mathfrak{S})$, cubes in $\mathcal{J}(\mathfrak{S})$ are pairwise disjoint by `dyadicproperty`. The same applies to $\mathcal{L}(\mathfrak{S})$.If $x \in \bigcup_{I \in \mathcal{D}} I$, then there exists by `coverdyadic` a cube $I \in \mathcal{D}$ with $x \in I$ and $s(I) = -S$. Then $I \in \mathcal{J}_0(\mathfrak{S})$. There exists an inclusion maximal cube in $\mathcal{J}_0(\mathfrak{S})$ containing I. This cube contains x and is contained in $\mathcal{J}(\mathfrak{S})$. This shows one inclusion in `eq-J-partition`, the other one follows from $\mathcal{J}(\mathfrak{S}) \subset \mathcal{D}$.The proof of the two inclusions in `eq-L-partition` is similar.

For a finite collection of pairwise disjoint cubes $\mathcal{C}$, define the projection operator

$$P_{\mathcal{C}} f(x) := \sum_{J \in \mathcal{C}} \mathbf{1}_J(x) \frac{1}{\mu(J)} \int_J f(y) \, \mathrm{d}\mu(y).$$

Given a scale $-S \leq s \leq S$ and a point $x \in \bigcup_{I \in \mathcal{D}, s(I)=s} I$, there exists a unique cube in $\mathcal{D}$ of scale s containing x by `coverdyadic`. We denote it by $I_s(x)$. Define for $\vartheta \in \Theta$ the nontangential maximal operator

$$T_{\mathcal{N}}^{\vartheta} f(x) := \sup_{-S \leq s_1 < S} \sup_{x' \in I_{s_1}(x)} \sup_{\substack{s_1 \leq s_2 \leq S \\ D^{s_2-1} \leq R_Q(\vartheta, x')}} \left| \sum_{s=s_1}^{s_2} \int K_s(x', y) f(y) \, \mathrm{d}\mu(y) \right|.$$

Define for each $\mathfrak{u} \in \mathfrak{U}$ the auxiliary operator

$$S_{1,\mathfrak{u}} f(x)$$

$$:= \sum_{I \in \mathcal{D}} \mathbf{1}_I(x) \sum_{\substack{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u})) \\ J \subset B(c(I), 16D^{s(I)}) \\ s(J) \leq s(I)}} \frac{D^{(s(J)-s(I))/a}}{\mu(B(c(I), 16D^{s(I)}))} \int_J |f(y)| \, \mathrm{d}\mu(y).$$

Define also the collection of balls

$$\mathcal{B} = \{B(c(I), 2^s D^{s(I)+t}) \ : \ I \in \mathcal{D}, 0 \leq s \leq S + 5, 0 \leq t \leq 2S + 3\}.$$

The following pointwise estimate for operators associated to sets $\mathfrak{T}(\mathfrak{u})$ is the main result of this subsection.
Lemma 1.7.1.3

Let $\mathfrak{u} \in \mathfrak{U}$ and $L \in \mathcal{L}(\mathfrak{T}(\mathfrak{u}))$. Let $x, x' \in L$. Then for all bounded functions f with bounded support

$$\left| \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}}[e(-\mathcal{Q}(\mathfrak{u})) f](x) \right|$$

$$\leq 2^{129a^3} (M_{\mathcal{B},1} + S_{1,\mathfrak{u}}) P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} |f|(x') + |T_{\mathcal{N}}^{\mathcal{Q}(\mathfrak{u})} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} f(x')|.$$

Proof for Lemma 1.7.1.3

Proof. By `definetp`, if $T_{\mathfrak{p}}[e(-\mathcal{Q}(\mathfrak{u}))f](x) \neq 0$, then $x \in E(\mathfrak{p})$. Combining this with $|e(\mathcal{Q}(\mathfrak{u})(x))| = 1$, we obtain

$$| \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}}[e(-\mathcal{Q}(\mathfrak{u}))f](x)|$$

$$= \left| \sum_{s \in \sigma(\mathfrak{u},x)} \int e(-\mathcal{Q}(\mathfrak{u})(y) + Q(x)(y) + \mathcal{Q}(\mathfrak{u})(x) - Q(x)(x)) \times K_s(x,y) f(y) \, \mathrm{d}\mu(y) \right|.$$

Using the triangle inequality, we bound this by the sum of three terms:

$$\leq \left| \sum_{s \in \sigma(\mathfrak{u},x)} \int (e(-\mathcal{Q}(\mathfrak{u})(y) + Q(x)(y) + \mathcal{Q}(\mathfrak{u})(x) - Q(x)(x)) - 1) \times K_s(x,y) f(y) \, \mathrm{d}\mu(y) \right|$$

$$+ \left| \sum_{s \in \sigma(\mathfrak{u},x)} \int K_s(x,y) P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} f(y) \, \mathrm{d}\mu(y) \right|$$

$$+ \left| \sum_{s \in \sigma(\mathfrak{u},x)} \int K_s(x,y) (f(y) - P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} f(y)) \, \mathrm{d}\mu(y) \right|.$$

The proof is completed using the bounds for these three terms proven respectively in Lemma 1.7.1.4, Lemma 1.7.1.5 and Lemma 1.7.1.6.

Lemma 1.7.1.4

For all $\mathfrak{u} \in \mathfrak{U}$, all $L \in \mathcal{L}(\mathfrak{T}(\mathfrak{u}))$, all $x, x' \in L$ and all bounded f with bounded support, we have the estimate `eq-term-A`:

$$10 \cdot 2^{104a^3} M_{\mathcal{B},1} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} |f|(x').$$

Proof for Lemma 1.7.1.4

Proof. Let $s \in \sigma(\mathfrak{u}, x)$. If $x, y \in X$ are such that $K_s(x,y) \neq 0$, then, by `supp-Ks`, we have $\rho(x,y) \leq 1/2D^s$. By 1-Lipschitz continuity of the function $t \mapsto \exp(it) = e(t)$ and the property `osccontrol` of the metrics d_B, it follows that

$$|e(-\mathcal{Q}(\mathfrak{u})(y) + Q(x)(y) + \mathcal{Q}(\mathfrak{u})(x) - Q(x)(x)) - 1| \leq d_{B(x,1/2D^s)}(\mathcal{Q}(\mathfrak{u}), Q(x)).$$

Let $\mathfrak{p}_s \in \mathfrak{T}(\mathfrak{u})$ be a tile with $\mathsf{s}(\mathfrak{p}_s) = s$ and $x \in E(\mathfrak{p}_s)$, and let $\mathfrak{p}'$ be a tile with $\mathsf{s}(\mathfrak{p}') = \overline{\sigma}(\mathfrak{u}, x)$ and $x \in E(\mathfrak{p}')$. Using the monotonicity property `monotonedb`, the doubling property `firstdb` repeatedly, the definition of $d_{\mathfrak{p}}$ and Lemma 1.2.1.2, we can bound the previous display by

$$d_{B(x,4D^s)}(\mathcal{Q}(\mathfrak{u}), Q(x)) \le 2^{4a} d_{\mathfrak{p}_s}(\mathcal{Q}(\mathfrak{u}), Q(x)) \le 2^{4a} 2^{s-\overline{\sigma}(\mathfrak{u},x)} d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{u}), Q(x)).$$

Since $\mathcal{Q}(\mathfrak{u}) \in B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), 4)$ by `forest1` and $Q(x) \in \Omega(\mathfrak{p}') \subset B_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), 1)$ by `eq-freq-comp-ball`, this is estimated by

$$\le 5 \cdot 2^{4a} 2^{s-\overline{\sigma}(\mathfrak{u},x)}.$$

Using `eq-Ks-size`, it follows that `eq-term-A` is bounded by

$$5 \cdot 2^{103a^3} \sum_{s \in \sigma(x)} 2^{s-\overline{\sigma}(\mathfrak{u},x)} \frac{1}{\mu(B(x, D^s))} \int_{B(x,0.5D^s)} |f(y)| \, \mathrm{d}\mu(y).$$

By `eq-J-partition`, the collection $\mathcal{J}$ is a partition of $\bigcup_{I \in \mathcal{D}} I$, so this is estimated by

$$5 \cdot 2^{103a^3} \sum_{s \in \sigma(x)} 2^{s-\overline{\sigma}(\mathfrak{u},x)} \frac{1}{\mu(B(x, D^s))} \sum_{\substack{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u})) \\ J \cap B(x,0.5D^s) \neq \emptyset}} \int_J |f(y)| \, \mathrm{d}\mu(y).$$

This expression does not change if we replace $|f|$ by $P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}|f|$.Let $J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))$ with $B(x, 0.5D^s) \cap J \neq \emptyset$. By the triangle inequality and since $x \in E(\mathfrak{p}_s) \subset B(\mathsf{c}(\mathfrak{p}_s), 4D^s)$, it follows that $B(\mathsf{c}(\mathfrak{p}_s), 4.5D^s) \cap J \neq \emptyset$. If $s(J) \ge s$ and $s(J) > -S$, then it follows from the triangle inequality, `eq-vol-sp-cube` and `defineD` that $\mathcal{I}(\mathfrak{p}_s) \subset B(c(J), 100D^{s(J)+1})$, contradicting $J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))$. Thus $s(J) \le s - 1$ or $s(J) = -S$. If $s(J) = -S$ and $s(J) > s - 1$, then $s = -S$. Thus we always have $s(J) \le s$. It then follows from the triangle inequality and `eq-vol-sp-cube` that $J \subset B(\mathsf{c}(\mathfrak{p}_s), 16D^s)$.Thus we can continue our chain of estimates with

$$5 \cdot 2^{103a^3} \sum_{s \in \sigma(x)} 2^{s-\overline{\sigma}(\mathfrak{u},x)} \frac{1}{\mu(B(x, D^s))} \int_{B(\mathsf{c}(\mathfrak{p}_s),16D^s)} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}|f(y)| \, \mathrm{d}\mu(y).$$

We have $B(\mathsf{c}(\mathfrak{p}_s), 16D^s)) \subset B(x, 32D^s)$, by `eq-vol-sp-cube` and the triangle inequality, since $x \in \mathcal{I}(\mathfrak{p}_s)$. Combining this with the doubling property `doublingx`, we obtain

$$\mu(B(\mathsf{c}(\mathfrak{p}_s), 16D^s)) \le 2^{5a} \mu(B(x, D^s)).$$

Since $a \ge 4$, it follows that `eq-term-A` is bounded by

$$5 \cdot 2^{103a^3} \sum_{s \in \sigma(x)} 2^{s-\overline{\sigma}(\mathfrak{u},x)} \frac{1}{\mu(B(\mathsf{c}(\mathfrak{p}_s), 16D^s))} \int_{B(\mathsf{c}(\mathfrak{p}_s),16D^s)} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}|f(y)| \, \mathrm{d}\mu(y).$$

Since $L \in \mathcal{L}(\mathfrak{T}(\mathfrak{u}))$ and $x \in L \cap \mathcal{I}(\mathfrak{p}_s)$, we have $s(L) \leq \mathsf{s}(\mathfrak{p}_s)$. It follows by `dyadicproperty` that $L \subset \mathcal{I}(\mathfrak{p}_s)$, in particular $x' \in \mathcal{I}(\mathfrak{p}_s) \subset B(\mathsf{c}(\mathfrak{p}_s), 16D^s)$. Thus

$$\leq 5 \cdot 2^{104a^3} \sum_{s \in \sigma(x)} 2^{s-\overline{\sigma}(\mathfrak{u},x)} M_{\mathcal{B},1} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} |f|(x')$$

$$\leq 10 \cdot 2^{104a^3} M_{\mathcal{B},1} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} |f|(x').$$

This completes the estimate for term `eq-term-A`.

Lemma 1.7.1.5

For all $\mathfrak{u} \in \mathfrak{U}$, all $L \in \mathcal{L}(\mathfrak{T}(\mathfrak{u}))$, all $x, x' \in L$ and all bounded f with bounded support, we have

$$\left| \sum_{s \in \sigma(\mathfrak{u},x)} \int K_s(x,y) P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} f(y) \, \mathrm{d}\mu(y) \right| \leq T_{\mathcal{N}}^{\mathcal{Q}(\mathfrak{u})} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} f(x').$$

Proof. Let $s_1 = \underline{\sigma}(\mathfrak{u}, x)$. By definition, there exists a tile $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ with $\mathsf{s}(\mathfrak{p}) = s_1$ and $x \in E(\mathfrak{p})$. Then $x \in \mathcal{I}(\mathfrak{p}) \cap L$. By `dyadicproperty` and the definition of $\mathcal{L}(\mathfrak{T}(\mathfrak{u}))$, it follows that $L \subset \mathcal{I}(\mathfrak{p})$, so $x \in I_{s_1}(x')$.
Proof for Lemma 1.7.1.5

Next, let $s_2 = \overline{\sigma}(\mathfrak{u}, x)$ and let $\mathfrak{p}' \in \mathfrak{T}(\mathfrak{u})$ with $\mathsf{s}(\mathfrak{p}') = s_2$ and $x \in E(\mathfrak{p}')$. Since $\mathfrak{p}' \in \mathfrak{T}(\mathfrak{u})$, we have $4\mathfrak{p}' \lesssim \mathfrak{u}$. Since $Q(x) \in \Omega(\mathfrak{p}')$, it follows that

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{u}), Q(x)) \leq 5.$$

Applying the doubling property `firstdb` five times, we obtain

$$d_{B(c(\mathfrak{p}), 8D^{s_2})}(\mathcal{Q}(\mathfrak{u}), Q(x)) \leq 5 \cdot 2^{5a}.$$

By the triangle inequality, we have $B(x, D^{s_2}) \subset B(c(\mathfrak{p}), 8D^{s_2})$, so by `monotonedb`

$$d_{B(x, D^{s_2})}(\mathcal{Q}(\mathfrak{u}), Q(x)) \leq 5 \cdot 2^{5a}.$$

Finally, by applying `seconddb` $100a$ times, we obtain

$$d_{B(x, D^{s_2-1})}(\mathcal{Q}(\mathfrak{u}), Q(x)) \leq 5 \cdot 2^{-95a} < 1.$$

Consequently, $D^{s_2-1} < R_Q(\mathcal{Q}(\mathfrak{u}), x)$. The lemma now follows from the definition of $T_{\mathcal{N}}$.

Lemma 1.7.1.6

For all $\mathfrak{u} \in \mathfrak{U}$, all $L \in \mathcal{L}(\mathfrak{T}(\mathfrak{u}))$, all $x, x' \in L$ and all bounded f with bounded support, we have

$$\left| \sum_{s \in \sigma(\mathfrak{u}, x)} \int K_s(x,y)(f(y) - P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} f(y)) \, \mathrm{d}\mu(y) \right|$$

$$\leq 2^{128a^3} S_{1,\mathfrak{u}} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} |f|(x').$$

Proof for Lemma 1.7.1.6

Proof. We have for $J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))$

$$\int_J K_s(x,y)(1 - P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}) f(y) \, \mathrm{d}\mu(y)$$

$$= \int_J \frac{1}{\mu(J)} \int_J K_s(x,y) - K_s(x,z) \, \mathrm{d}\mu(z) \, f(y) \, \mathrm{d}\mu(y).$$

By `eq-Ks-smooth` and `eq-vol-sp-cube`, we have for $y, z \in J$

$$|K_s(x,y) - K_s(x,z)| \leq \frac{2^{127a^3}}{\mu(B(x, D^s))} \left(\frac{8D^{s(J)}}{D^s} \right)^{1/a}.$$

Suppose that $s \in \sigma(\mathfrak{u}, x)$. If $K_s(x,y) \neq 0$ for some $y \in J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))$ then, by `supp-Ks`, $y \in B(x, 0.5D^s) \cap J \neq \emptyset$. Let $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ with $\mathrm{s}(\mathfrak{p}) = s$ and $x \in E(\mathfrak{p})$. Then $B(\mathrm{c}(\mathfrak{p}_s), 4.5D^s) \cap J \neq \emptyset$ by the triangle inequality. If $s(J) \geq s$ and $s(J) > -S$, then it follows from the triangle inequality, `eq-vol-sp-cube` and `defineD` that $\mathcal{I}(\mathfrak{p}) \subset B(c(J), 100D^{s(J)+1})$, contradicting $J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))$. Thus $s(J) \leq s - 1$ or $s(J) = -S$. If $s(J) = -S$ and $s(J) > s - 1$, then $s = -S$. So in both cases, $s(J) \leq s$. It then follows from the triangle inequality and `eq-vol-sp-cube` that $J \subset B(x, 16D^s)$.Thus, we can estimate `eq-term-C` by

$$2^{127a^3+3/a} \sum_{\mathfrak{p} \in \mathfrak{T}} \frac{\mathbf{1}_{E(\mathfrak{p})}(x)}{\mu(B(x, D^{\mathrm{s}(\mathfrak{p})}))} \sum_{\substack{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u})) \\ J \subset B(x, 16D^{\mathrm{s}(\mathfrak{p})}) \\ s(J) \leq \mathrm{s}(\mathfrak{p})}} D^{(s(J) - \mathrm{s}(\mathfrak{p}))/a} \int_J |f|.$$

$$= 2^{127a^3+3/a} \sum_{I \in \mathcal{D}} \sum_{\substack{\mathfrak{p} \in \mathfrak{T} \\ \mathcal{I}(\mathfrak{p}) = I}} \frac{\mathbf{1}_{E(\mathfrak{p})}(x)}{\mu(B(x, D^{s(I)}))} \sum_{\substack{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u})) \\ J \subset B(x, 16D^{s(I)}) \\ s(J) \leq s(I)}} D^{(s(J) - s(I))/a} \int_J |f|.$$

By `eq-dis-freq-cover` and `defineep`, the sets $E(\mathfrak{p})$ for tiles $\mathfrak{p}$ with $\mathcal{I}(\mathfrak{p}) = I$ are pairwise disjoint. It follows from the definition of $\mathcal{L}(\mathfrak{T}(\mathfrak{u}))$ that $x \in \mathcal{I}(\mathfrak{p})$ if and only if $x' \in \mathcal{I}(\mathfrak{p})$, thus we can estimate the sum over $\mathbf{1}_{E(\mathfrak{p})}(x)$ by $\mathbf{1}_I(x')$.If $x \in E(\mathfrak{p})$ then in particular

$x \in \mathcal{I}(\mathfrak{p})$, so by `eq-vol-sp-cube` $B(c(I), 16D^{s(I)}) \subset B(x, 32D^{s(I)})$. By the doubling property `doublingx`

$$\mu(B(c(I), 16D^{s(I)})) \leq 2^{5a}\mu(B(x, D^{s(I)})).$$

Since $a \geq 4$ we can continue our estimate with

$$\leq 2^{128a^3} \sum_{I \in \mathcal{D}} \frac{\mathbf{1}_I(x')}{\mu(B(c(I), 16D^{s(I)}))} \sum_{\substack{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u})) \\ J \subset B(x, 16D^{s(I)}) \\ s(J) \leq s(I)}} D^{(s(J)-s(I))/a} \int_J |f|$$

$$= 2^{128a^3} S_{1,\mathfrak{u}} P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}|f|(x').$$

This completes the proof.

An auxiliary L^2 tree estimate

In this subsection we prove the following estimate on L^2 for operators associated to trees.

Lemma 1.7.2.1

Let $\mathfrak{u} \in \mathfrak{U}$. Then we have for all f, g bounded with bounded support

$$\left| \int_X \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} \bar{g}(y) T_{\mathfrak{p}} f(y) \, \mathrm{d}\mu(y) \right|$$

$$\leq 2^{130a^3} \|P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}|f|\|_2 \|P_{\mathcal{L}(\mathfrak{T}(\mathfrak{u}))}|g|\|_2.$$

Below, we deduce Lemma 1.7.2.1 from Lemma 1.7.1.3 and the following estimates for the operators in Lemma 1.7.1.3.

Lemma 1.7.2.2

For all bounded f with bounded support and all $\vartheta \in \Theta$

$$\|T_{\mathcal{N}}^{\vartheta} f\|_2 \leq 2^{102a^3} \|f\|_2.$$

Lemma 1.7.2.3

For all $\mathfrak{u} \in \mathfrak{U}$ and all bounded functions f with bounded support

$$\|S_{1,\mathfrak{u}} f\|_2 \leq 2^{12a} \|f\|_2.$$

Proof for Lemma 1.7.2.1

Proof of Lemma 1.7.2.1. Let $L \in \mathcal{L}(\mathfrak{T}(\mathfrak{u}))$. Let $b(x')$ denote the right-hand side of Lemma 1.7.1.3. Apply this lemma to $e(\mathcal{Q}(\mathfrak{u}))f$, to obtain for all $y, x' \in L$

$$\left|\sum_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f(y)\right| \le b(x').$$

Hence, by taking an infimum, we have for $y \in L$

$$\left|\sum_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f(y)\right| \le \inf_{x'\in L} b(x').$$

Integrating this estimate yields

$$\int_L |g(y)| \left|\sum_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f(y)\right| \mathrm{d}\mu(y)$$

$$\le \int_L |g(y)| \times \inf_{x'\in L} b(x') \,\mathrm{d}\mu(y)$$

$$= \int_L P_{\mathcal{L}(\mathfrak{T}(\mathfrak{u}))}|g|(y) \times \inf_{x'\in L} b(x') \,\mathrm{d}\mu(y)$$

$$\le \int_L P_{\mathcal{L}(\mathfrak{T}(\mathfrak{u}))}|g|(y) \times b(y) \,\mathrm{d}\mu(y).$$

By `definetp`, we have $T_{\mathfrak{p}} f = \mathbf{1}_{\mathcal{I}(\mathfrak{p})} T_{\mathfrak{p}} f$ for all $\mathfrak{p} \in \mathfrak{P}$, so

$$\left|\int \bar{g}(y) \sum_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f(y) \,\mathrm{d}\mu(y)\right| = \left|\int_{\bigcup_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} \mathcal{I}(\mathfrak{p})} \bar{g}(y) \sum_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f(y) \,\mathrm{d}\mu(y)\right|.$$

Since $\mathcal{L}(\mathfrak{T}(\mathfrak{u}))$ partitions $\bigcup_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} \mathcal{I}(\mathfrak{p})$ by Lemma 1.7.1.2, we get from the triangle inequality

$$\le \sum_{L\in\mathcal{L}(\mathfrak{T}(\mathfrak{u}))} \int_L |g(y)| \left|\sum_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f(y)\right| \mathrm{d}\mu(y)$$

which by the above computation is bounded by

$$\sum_{L\in\mathcal{L}(\mathfrak{T}(\mathfrak{u}))} \int_L P_{\mathcal{L}(\mathfrak{T}(\mathfrak{u}))}|g|(y) \times b(y) \,\mathrm{d}\mu(y)$$

$$= \int_X P_{\mathcal{L}(\mathfrak{T}(\mathfrak{u}))}|g|(y) \times b(y) \,\mathrm{d}\mu(y).$$

Applying Cauchy-Schwarz, this is bounded by $\|P_{\mathcal{L}(\mathfrak{T}(\mathfrak{u}))}|g|\|_2 \times \|b\|_2$. By Minkowski's inequality, Theorem 1.2.6, Lemma 1.7.2.2 and Lemma 1.7.2.3, $\|b\|_2$ is at most

$$2^{129a^3}(2^{2a+1} + 2^{12a})\|P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}|f|\|_2 + 2^{103a^3}\|P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}[e(\mathcal{Q}(\mathfrak{u}))f]\|_2.$$

By the triangle inequality we have for all $x \in X$ that $|P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}[e(\mathcal{Q}(\mathfrak{u}))f]|(x) \le P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}|f|(x)$, thus we can further estimate the above by

$$(2^{129a^3}(2^{2a+1} + 2^{12a}) + 2^{103a^3})\|P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))}|f|\|_2.$$

This completes the proof since $a \ge 4$.

Now we prove the two auxiliary lemmas. We begin with the nontangential maximal operator $T_{\mathcal{N}}$.

Proof for Lemma 1.7.2.2

Proof of Lemma 1.7.2.2. Fix s_1, s_2. By `eq-psisum` we have for all $x \in (0, \infty)$

$$\sum_{s=s_1}^{s_2} \psi(D^{-s}x) = 1 - \sum_{s<s_1} \psi(D^{-s}x) - \sum_{s>s_1} \psi(D^{-s}x).$$

Since ψ is supported in $[\frac{1}{4D}, \frac{1}{2}]$, the two sums on the right hand side are zero for all $x \in [\frac{1}{2}D^{s_1-1}, \frac{1}{4}D^{s_2-1}]$, hence

$$x \in [\frac{1}{2}D^{s_1-1}, \frac{1}{4}D^{s_2}] \implies \sum_{s=s_1}^{s_2} \psi(D^{-s}x) = 1.$$

Since ψ is supported in $[\frac{1}{4D}, \frac{1}{2}]$, we further have

$$x \notin [\frac{1}{4}D^{s_1-1}, \frac{1}{2}D^{s_2}] \implies \sum_{s=s_1}^{s_2} \psi(D^{-s}x) = 0.$$

Finally, since $\psi \ge 0$ and $\sum_{s\in\mathbb{Z}} \psi(D^{-s}x) = 1$, we have for all x

$$0 \le \sum_{s=s_1}^{s_2} \psi(D^{-s}x) \le 1.$$

Let $x' \in I_{s_1}(x)$ and suppose that $D^{s_2-1} \le R_Q(\vartheta, x')$. By the triangle inequality and `eq-vol-sp-cube`, it holds that $\rho(x, x') \le 8D^{s_1}$. We have

$$\left|\sum_{s=s_1}^{s_2} \int K_s(x', y) f(y)\, \mathrm{d}\mu(y)\right|$$

$$= \left| \int \sum_{s=s_1}^{s_2} \psi(D^{-s}\rho(x',y)) K(x',y) f(y) \, \mathrm{d}\mu(y) \right|$$

$$\leq \left| \int_{8D^{s_1} < \rho(x',y) \leq \frac{1}{4}D^{s_2-1}} K(x',y) f(y) \, \mathrm{d}\mu(y) \right|$$

$$+ \int_{\frac{1}{4}D^{s_1-1} \leq \rho(x',y) \leq 8D^{s_1}} |K(x',y)||f(y)| \, \mathrm{d}\mu(y)$$

$$+ \int_{\frac{1}{4}D^{s_2-1} \leq \rho(x',y) \leq \frac{1}{2}D^{s_2}} |K(x',y)||f(y)| \, \mathrm{d}\mu(y).$$

The first term `eq-sharp-trunc-term` is at most $2T_Q^\vartheta f(x)$, using with $R_1 := 8D^{s_1}$, $R_2 := \frac{1}{4}D^{s_2-1}$ and $R_1 < R_2 < R_Q(\vartheta, x')$ the triangle inequality in the form

$$\left| \int_{R_1 < \rho(x',y) \leq R_2} K(x',y) f(y) \, \mathrm{d}\mu(y) \right|$$

$$\leq \left| \int_{R_1 < \rho(x',y) < R_Q(\vartheta,x')} K(x',y) f(y) \, \mathrm{d}\mu(y) \right|$$

$$+ \left| \int_{R_2 < \rho(x',y) < R_Q(\vartheta,x')} K(x',y) f(y) \, \mathrm{d}\mu(y) \right|.$$

The other two terms will be estimated by the finitary maximal function from Theorem 1.2.6. For the second term `eq-lower-bound-term` we use `eqkernel-size` which implies that for all y with $\rho(x',y) \geq \frac{1}{4}D^{s_1-1}$, we have

$$|K(x',y)| \leq \frac{2^{a^3}}{\mu(B(x', \frac{1}{4}D^{s_1-1}))}.$$

Using $D = 2^{100a^2}$ and the doubling property `doublingx` $7 + 100a^2$ times estimates the last display by

$$\frac{2^{7a+101a^3}}{\mu(B(x', 32D^{s_1}))}.$$

By the triangle inequality and `eq-vol-sp-cube`, we have

$$B(x', 8D^{s_1}) \subset B(c(I_{s_1}(x)), 16D^{s(I_{s_1}(x))}) \subset B(x', 32D^{s_1}).$$

Combining this with `pf-nontangential-operator-bound-imeq`, we conclude that `eq-lower-bound-term` is at most

$$2^{7a+101a^3} M_{\mathcal{B},1} f(x).$$

For eq-upper-bound-term we argue similarly. We have for all y with $\rho(x', y) \geq \frac{1}{4}D^{s_2-1}$

$$|K(x', y)| \leq \frac{2^{a^3}}{\mu(B(x', \frac{1}{4}D^{s_2-1}))}.$$

Using the doubling property doublingx $7 + 100a^2$ times estimates the last display by

$$\frac{2^{7a+101a^3}}{\mu(B(x', 32D^{s_2}))}.$$

Note that by dyadicproperty we have $I_{s_1}(x) \subset I_{s_2}(x)$, in particular $x' \in I_{s_2}(x)$. By the triangle inequality and eq-vol-sp-cube, we have

$$B(x', 8D^{s_2}) \subset B(c(I_{s_2}(x)), 16D^{s(I_{s_2}(x))}) \subset B(x', 32D^{s_2}).$$

Combining this, eq-upper-bound-term is at most

$$2^{7a+101a^3} M_{\mathcal{B},1} f(x).$$

Using $a \geq 4$, taking a supremum over all $x' \in I_{s_1}(x)$ and then a supremum over all $-S \leq s_1 < s_2 \leq S$, we obtain

$$T_{\mathcal{N}} f(x) \leq 2T_Q^{\vartheta} f(x) + 2^{102a^3} M_{\mathcal{B},1} f(x).$$

The lemma now follows from assumption linnontanbound, Theorem 1.2.6 and $a \geq 4$.

We need the following lemma to prepare the L^2-estimate for the auxiliary operators $S_{1,\mathfrak{u}}$.
Lemma 1.7.2.4

For every cube $I \in \mathcal{D}$, there exist at most 2^{9a} cubes $J \in \mathcal{D}$ with $s(J) = s(I)$ and $B(c(I), 16D^{s(I)}) \cap B(c(J), 16D^{s(J)}) \neq \emptyset$.

Proof. Suppose that $B(c(I), 16D^{s(I)}) \cap B(c(J), 16D^{s(J)}) \neq \emptyset$ and $s(I) = s(J)$. Then $B(c(I), 33D^{s(I)}) \subset B(c(J), 128D^{s(J)})$. Hence by the doubling property doublingx

$$2^{9a}\mu(B(c(J), \frac{1}{4}D^{s(J)})) \geq \mu(B(c(I), 33D^{s(I)})),$$

and by the triangle inequality, $B(c(J), \frac{1}{4}D^{s(J)})$ is contained in $B(c(I), 33D^{s(I)})$.
Proof for Lemma 1.7.2.4

If $\mathcal{C}$ is any finite collection of cubes $J \in \mathcal{D}$ satisfying $s(J) = s(I)$ and

$$B(c(I), 16D^{s(I)}) \cap B(c(J), 16D^{s(J)}) \neq \emptyset,$$

then it follows from `eq-vol-sp-cube` and pairwise disjointedness of cubes of the same scale `dyadicproperty` that the balls $B(c(J), \frac{1}{4}D^{s(J)})$ are pairwise disjoint. Hence

$$\mu(B(c(I), 33D^{s(I)})) \geq \sum_{J \in \mathcal{C}} \mu(B(c(J), \frac{1}{4}D^{s(J)}))$$

$$\geq |\mathcal{C}| 2^{-9a} \mu(B(c(I), 33D^{s(I)})).$$

Since μ is doubling and $\mu \neq 0$, we have $\mu(B(c(I), 33D^{s(I)})) > 0$. The lemma follows after dividing by $2^{-9a}\mu(B(c(I), 33D^{s(I)}))$.

Now we can bound the operators $S_{1,\mathfrak{u}}$.

Proof for Lemma 1.7.2.3

Proof of Lemma 1.7.2.3. Note that by definition, $S_{1,\mathfrak{u}}f$ is a finite sum of indicator functions of cubes $I \in \mathcal{D}$ for each locally integrable f, and hence is bounded, has bounded support and is integrable. Let g be another function with the same three properties. Then $\bar{g}S_{1,\mathfrak{u}}f$ is integrable, and we have

$$\left| \int \bar{g}(y) S_{1,\mathfrak{u}} f(y) \,\mathrm{d}\mu(y) \right|$$

$$= \left| \sum_{I \in \mathcal{D}} \frac{1}{\mu(B(c(I), 16D^{s(I)}))} \int_I \bar{g}(y) \,\mathrm{d}\mu(y) \times \sum_{\substack{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u})) \,:\, J \subseteq B(c(I), 16D^{s(I)}) \\ s(J) \leq s(I)}} D^{(s(J)-s(I))/a} \int_J |f(y)| \,\mathrm{d}\mu(y) \right|$$

$$\leq \sum_{I \in \mathcal{D}} \frac{1}{\mu(B(c(I), 16D^{s(I)}))} \int_{B(c(I), 16D^{s(I)})} |g(y)| \,\mathrm{d}\mu(y) \times \sum_{\substack{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u})) \,:\, J \subseteq B(c(I), 16D^{s(I)}) \\ s(J) \leq s(I)}} D^{(s(J)-s(I))/a} \int_J |f(y)|$$

Changing the order of summation and using $J \subset B(c(I), 16D^{s(I)})$ to bound the first average integral by $M_{\mathcal{B},1}|g|(y)$ for any $y \in J$, we obtain

$$\leq \sum_{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))} \int_J |f(y)| M_{\mathcal{B},1}|g|(y) \,\mathrm{d}\mu(y) \sum_{\substack{I \in \mathcal{D} \,:\, J \subset B(c(I), 16D^{s(I)}) \\ s(I) \geq s(J)}} D^{(s(J)-s(I))/a}.$$

By Lemma 1.7.2.4, there are at most 2^{9a} cubes I at each scale satisfying the inclusion $J \subset B(c(I), 16D^{s(I)})$. Since $D^{-1/a} \leq \frac{1}{2}$, $(1 -$

$D^{-1/a})^{-1} \le 2$. Using this estimate for the sum of the geometric series, we conclude that `eq-boundary-operator-bound-1` is at most

$$2^{9a+1} \sum_{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))} \int_J |f(y)| M_{\mathcal{B},1}|g|(y)\, \mathrm{d}\mu(y).$$

The collection $\mathcal{J}$ is a partition of $\bigcup_{I \in \mathcal{D}} I$, so this equals

$$2^{9a+1} \int_X |f(y)| M_{\mathcal{B},1}|g|(y)\, \mathrm{d}\mu(y).$$

Using Cauchy-Schwarz and Theorem 1.2.6, we conclude

$$\left| \int \bar{g} S_{1,\mathfrak{u}} f \, \mathrm{d}\mu \right| \le 2^{12a} \|g\|_2 \|f\|_2.$$

The lemma now follows by choosing $g = S_{1,\mathfrak{u}} f$ and dividing on both sides by the finite $\|S_{1,\mathfrak{u}} f\|_2$.

The quantitative L^2 tree estimate

The main result of this subsection is the following quantitative bound for operators associated to trees, with decay in the densities dens_1 and dens_2.
Lemma 1.7.3.1

Let $\mathfrak{u} \in \mathfrak{U}$. Then for all bounded f with bounded support and bounded g supported on G we have

$$\left| \int_X \bar{g} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f \, \mathrm{d}\mu \right| \le 2^{181a^3} \, \mathrm{dens}_1(\mathfrak{T}(\mathfrak{u}))^{1/2} \|f\|_2 \|g\|_2.$$

If additionally $\mathrm{support}(f) \subseteq F$, then we have

$$\left| \int_X \bar{g} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f \, \mathrm{d}\mu \right| \le 2^{282a^3} \, \mathrm{dens}_1(\mathfrak{T}(\mathfrak{u}))^{1/2} \, \mathrm{dens}_2(\mathfrak{T}(\mathfrak{u}))^{1/2} \|f\|_2 \|g\|_2.$$

Below, we deduce this lemma from Lemma 1.7.2.1 and the following two estimates controlling the size of support of the operator and its adjoint.
Lemma 1.7.3.2

Let $\mathfrak{u} \in \mathfrak{U}$ and $L \in \mathcal{L}(\mathfrak{T}(\mathfrak{u}))$. Then

$$\mu(L \cap G \cap \bigcup_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} E(\mathfrak{p})) \le 2^{101a^3} \, \mathrm{dens}_1(\mathfrak{T}(\mathfrak{u})) \mu(L).$$

Lemma 1.7.3.3

Let $\mathfrak{u} \in \mathfrak{U}$ and $J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))$. Then

$$\mu(F \cap J) \leq 2^{201a^3} \operatorname{dens}_2(\mathfrak{T}(\mathfrak{u}))\mu(J).$$

Proof for Lemma 1.7.3.1

Proof of Lemma 1.7.3.1. Denote

$$\mathcal{E}(\mathfrak{u}) = \bigcup_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} E(\mathfrak{p}).$$

Then we have

$$\left| \int_X \bar{g} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f \, \mathrm{d}\mu \right| = \left| \int_X \overline{g \mathbf{1}_{\mathcal{E}(\mathfrak{u})}} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} (\mathbf{1}_{\mathcal{I}(\mathfrak{u})} f) \, \mathrm{d}\mu \right|.$$

By Lemma 1.7.2.1, this is bounded by

$$\leq 2^{130a^3} \|P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} |f|\|_2 \|P_{\mathcal{L}(\mathfrak{T}(\mathfrak{u}))} |\mathbf{1}_{\mathcal{E}(\mathfrak{u})} g|\|_2.$$

We bound the two factors separately. We have

$$\|P_{\mathcal{L}(\mathfrak{T}(\mathfrak{u}))} |\mathbf{1}_{\mathcal{E}(\mathfrak{u})} g|\|_2 = \left(\sum_{L \in \mathcal{L}(\mathfrak{T}(\mathfrak{u}))} \frac{1}{\mu(L)} \left(\int_{L \cap \mathcal{E}(\mathfrak{u})} |g(y)| \, \mathrm{d}\mu(y) \right)^2 \right)^{1/2}.$$

By Cauchy-Schwarz and Lemma 1.7.3.2 this is at most

$$\leq \left(\sum_{L \in \mathcal{L}(\mathfrak{T}(\mathfrak{u}))} 2^{101a^3} \operatorname{dens}_1(\mathfrak{T}(\mathfrak{u})) \int_{L \cap \mathcal{E}(\mathfrak{u})} |g(y)|^2 \, \mathrm{d}\mu(y) \right)^{1/2}.$$

Since cubes $L \in \mathcal{L}(\mathfrak{T}(\mathfrak{u}))$ are pairwise disjoint by Lemma 1.7.1.2, this is

$$\leq 2^{51a^3} \operatorname{dens}_1(\mathfrak{T}(\mathfrak{u}))^{1/2} \|g\|_2.$$

Similarly, we have

$$\|P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}))} |f|\|_2 = \left(\sum_{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))} \frac{1}{\mu(J)} \left(\int_J |f(y)| \, \mathrm{d}\mu(y) \right)^2 \right)^{1/2}.$$

By Cauchy-Schwarz, this is

$$\leq \left(\sum_{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))} \int_J |f(y)|^2 \, \mathrm{d}\mu(y) \right)^{1/2}.$$

Since cubes in $\mathcal{J}(\mathfrak{T}(\mathfrak{u}))$ are pairwise disjoint by Lemma 1.7.1.2, this is at most

$$\|f\|_2.$$

Combining `eq-both-factors-tree`, `eq-factor-L-tree` and `eq-factor-J-tree` gives `eq-cor-tree-est`.If $f \leq \mathbf{1}_F$ then $f = f\mathbf{1}_F$, so

$$\left(\sum_{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))} \frac{1}{\mu(J)} \left(\int_J |f(y)| \, \mathrm{d}\mu(y)\right)^2\right)^{1/2}$$

$$= \left(\sum_{J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}))} \frac{1}{\mu(J)} \left(\int_{J \cap F} |f(y)| \, \mathrm{d}\mu(y)\right)^2\right)^{1/2}.$$

We estimate as before, using now Lemma 1.7.3.3 and Cauchy-Schwarz, and obtain that this is

$$\leq 2^{101a^3} \operatorname{dens}_2(\mathfrak{T}(\mathfrak{u}))^{1/2} \|f\|_2.$$

Combining this with `eq-both-factors-tree` and `eq-factor-L-tree` gives `eq-cor-tree-est-F`.

Now we prove the two auxiliary estimates.

Proof for Lemma 1.7.3.2

Proof of Lemma 1.7.3.2. If the set on the right hand side is empty, then `eq-1density-estimate-tree` holds. If not, then there exists $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ with $L \cap \mathcal{I}(\mathfrak{p}) \neq \emptyset$.Continuing the proof of Lemma 1.7.3.2, suppose first that there exists such $\mathfrak{p}$ with $\mathrm{s}(\mathfrak{p}) \leq s(L)$. Then by `dyadicproperty` $\mathcal{I}(\mathfrak{p}) \subset L$, which gives by the definition of $\mathcal{L}(\mathfrak{T}(\mathfrak{u}))$ that $s(L) = -S$ and hence $L = \mathcal{I}(\mathfrak{p})$. Let $\mathfrak{q} \in \mathfrak{T}(\mathfrak{u})$ with $E(\mathfrak{q}) \cap L \neq \emptyset$. Since $s(L) = -S \leq \mathrm{s}(\mathfrak{q})$ it follows from `dyadicproperty` that $\mathcal{I}(\mathfrak{p}) = L \subset \mathcal{I}(\mathfrak{q})$. We have then by Lemma 1.2.1.2

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{q})) \leq d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{u})) + d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{q}), \mathcal{Q}(\mathfrak{u}))$$

$$\leq d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{u})) + d_{\mathfrak{q}}(\mathcal{Q}(\mathfrak{q}), \mathcal{Q}(\mathfrak{u})).$$

Using that $\mathfrak{p}, \mathfrak{q} \in \mathfrak{T}(\mathfrak{u})$ and `forest1`, this is at most 8. Using again the triangle inequality and Lemma 1.2.1.2, we obtain that for each $q \in B_{\mathfrak{q}}(\mathcal{Q}(\mathfrak{q}), 1)$

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), q) \leq d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{q})) + d_{\mathfrak{q}}(\mathcal{Q}(\mathfrak{q}), q) \leq 9.$$

Thus $L \cap G \cap E(\mathfrak{q}) \subset E_2(9, \mathfrak{p})$. We obtain

$$\mu(L \cap G \cap \bigcup_{\mathfrak{q} \in \mathfrak{T}(\mathfrak{u})} E(\mathfrak{q})) \leq \mu(E_2(9, \mathfrak{p})).$$

By the definition of dens_1, this is bounded by

$$9^a \,\mathrm{dens}_1(\mathfrak{T}(\mathfrak{u}))\mu(\mathcal{I}(\mathfrak{p})) = 9^a \,\mathrm{dens}_1(\mathfrak{T}(\mathfrak{u}))\mu(L).$$

Since $a \geq 4$, `eq-1density-estimate-tree` follows in this case.Now suppose that for each $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ with $L \cap E(\mathfrak{p}) \neq \emptyset$, we have $\mathsf{s}(\mathfrak{p}) > s(L)$. Since there exists at least one such $\mathfrak{p}$, there exists in particular at least one cube $L'' \in \mathcal{D}$ with $L \subset L''$ and $s(L'') > s(L)$. By `coverdyadic`, there exists $L' \in \mathcal{D}$ with $L \subset L'$ and $s(L') = s(L) + 1$. By the definition of $\mathcal{L}(\mathfrak{T}(\mathfrak{u}))$ there exists a tile $\mathfrak{p}'' \in \mathfrak{T}(\mathfrak{u})$ with $\mathcal{I}(\mathfrak{p}'') \subset L'$.It suffices to show that there exists a tile $\mathfrak{p}' \in \mathfrak{P}(\mathfrak{T}(\mathfrak{u}))$ with $\mathcal{I}(\mathfrak{p}') = L'$, $d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \mathcal{Q}(\mathfrak{u})) < 4$ and $9\mathfrak{p}'' \lesssim 9\mathfrak{p}'$. For then, let $\mathfrak{q} \in \mathfrak{T}(\mathfrak{u})$ with $L \cap E(\mathfrak{q}) \neq \emptyset$. As shown above, this implies $\mathsf{s}(\mathfrak{q}) \geq s(L')$, so by `dyadicproperty` $L' \subset \mathcal{I}(\mathfrak{q})$. If $q \in B_{\mathfrak{q}}(\mathcal{Q}(\mathfrak{q}), 1)$, then by a similar calculation as above, using the triangle inequality, Lemma 1.2.1.2 and `forest1`, we obtain

$$d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), q) \leq d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \mathcal{Q}(\mathfrak{q})) + d_{\mathfrak{q}}(\mathcal{Q}(\mathfrak{q}), q) \leq 9.$$

Thus $L \cap G \cap E(\mathfrak{q}) \subset E_2(9, \mathfrak{p}')$. We deduce using the definition `definedens1` of dens_1

$$\mu(L \cap G \cap \bigcup_{\mathfrak{q} \in \mathfrak{T}(\mathfrak{u})} E(\mathfrak{q})) \leq \mu(E_2(9, \mathfrak{p}')) \leq 9^a \,\mathrm{dens}_1(\mathfrak{T}(\mathfrak{u}))\mu(L').$$

Using the doubling property `doublingx`, `eq-vol-sp-cube`, and $a \geq 4$ this is estimated by

$$9^a 2^{100a^3+5a} \,\mathrm{dens}_1(\mathfrak{T}(\mathfrak{u}))\mu(L) \leq 2^{101a^3} \,\mathrm{dens}_1(\mathfrak{T}(\mathfrak{u}))\mu(L).$$

To show existence of $\mathfrak{p}'$ with the given properties, if $\mathcal{I}(\mathfrak{p}'') = L'$ we can take $\mathfrak{p}' = \mathfrak{p}''$, which satisfies the distance property by `forest1` and the other properties trivially. Otherwise, let $\mathfrak{p}'$ be the unique tile such that $\mathcal{I}(\mathfrak{p}') = L'$ and such that $\Omega(\mathfrak{u}) \cap \Omega(\mathfrak{p}') \neq \emptyset$. Since $\mathcal{I}(\mathfrak{p}') \subset \mathcal{I}(\mathfrak{p})$ and $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$, we have $\mathfrak{p}' \in \mathfrak{P}(\mathfrak{T}(\mathfrak{u}))$. Since by `forest1` $\mathsf{s}(\mathfrak{p}') = s(L') \leq \mathsf{s}(\mathfrak{p}) < \mathsf{s}(\mathfrak{u})$, we have by `dyadicproperty` and `eq-freq-dyadic` that $\Omega(\mathfrak{u}) \subset \Omega(\mathfrak{p}')$, and hence the distance property. $9\mathfrak{p}'' \lesssim 9\mathfrak{p}'$ follows by the triangle inequality, `forest1`, Lemma 1.2.1.2 and `eq-freq-comp-ball`. This completes the proof.

Proof for Lemma 1.7.3.3

Proof of Lemma 1.7.3.3. We prove the inequality with the constant 2^{201a^3} replaced by 2^{200a^3+14a}; this is stronger because $a \geq 4$. It suffices to show the existence of a tile $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ and an $r \geq 4D^{\mathsf{s}(\mathfrak{p})}$ such that $J \subset B(\mathsf{c}(\mathfrak{p}), r)$ and $\mu(B(\mathsf{c}(\mathfrak{p}), r)) \leq 2^{200a^3+14a}\mu(J)$, because then it follows from the definition `definedens2` of dens_2 that

$$\mu(F \cap J) \leq \mu(F \cap B(\mathsf{c}(\mathfrak{p}), r))$$

$$\leq \mathrm{dens}_2(\mathfrak{T}(\mathfrak{u}))\mu(B(\mathtt{c}(\mathfrak{p}), r)) \leq 2^{200a^3+14a}\,\mathrm{dens}_2(\mathfrak{T}(\mathfrak{u}))\mu(J).$$

In particular, these criteria are satisfied, with $r = 4D^{\mathrm{s}(\mathfrak{p})}$, by any $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ such that $J \subseteq B(\mathtt{c}(\mathfrak{p}, 4D^{\mathrm{s}(\mathfrak{p})}))$ and $\mu(\mathcal{I}(\mathfrak{p})) \leq 2^{100a^3+10a}\mu(J)$, because then by the doubling property `doublingx`

$$\mu(B(\mathtt{c}(\mathfrak{p}), 4D^{\mathrm{s}(\mathfrak{p})})) \leq 2^{4a}\mu(B(\mathtt{c}(\mathfrak{p}), D^{\mathrm{s}(\mathfrak{p})}/4))$$

$$\leq 2^{4a}\mu(\mathcal{I}(\mathfrak{p})) \leq 2^{100a^3+14a}\mu(J).$$

Suppose first that $s(J) = S$. Then $J = I_0$, so `subsetmaxcube` and the fact that $J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u})) \subseteq \mathcal{J}_0(\mathfrak{T}(\mathfrak{u}))$ imply that $s(J) = -S$. Thus $S = 0$. It follows that J is the only dyadic cube, so any $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ has $\mathcal{I}(\mathfrak{p}) = J$, and therefore satisfies $J \subseteq B(\mathtt{c}(\mathfrak{p}, 4D^{\mathrm{s}(\mathfrak{p})}))$ and $\mu(\mathcal{I}(\mathfrak{p})) \leq 2^{100a^3+10a}\mu(J)$.It remains to consider the case $s(J) < S$. Then, by `coverdyadic` and `dyadicproperty`, there exists some cube $J' \in \mathcal{D}$ with $s(J') = s(J) + 1$ and $J \subset J'$. By definition of $\mathcal{J}(\mathfrak{T}(\mathfrak{u}))$ there exists some $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ such that $\mathcal{I}(\mathfrak{p}) \subset B(c(J'), 100D^{s(J')+1})$.Since $c(J) \in J \subset J' \subset B(c(J'), 4D^{s(J')})$, the triangle inequality, $s(J') = s(J) + 1$ and $D = 2^{100a^2}$ imply

$$B(c(J'), 204D^{s(J')+1}) \subset B(c(J), 204D^{s(J')+1}+4D^{s(J')}) \subset B(c(J), 2^8 D^{s(J)+2}).$$

From the doubling property `doublingx`, $D = 2^{100a^2}$ and `eq-vol-sp-cube`, we obtain

$$\mu(B(c(J'), 204D^{s(J')+1})) \leq 2^{200a^3+10a}\mu(J).$$

If $J \subset B(\mathtt{c}(\mathfrak{p}), 4D^{\mathrm{s}(\mathfrak{p})})$, then we need only check that $\mu(\mathcal{I}(\mathfrak{p})) \leq 2^{100a^3+10a}\mu(J)$. This follows immediately from $\mathcal{I}(\mathfrak{p}) \subset B(c(J'), 100D^{s(J')+1})$ and `measure-comparison`.From now on we assume $J \not\subset B(\mathtt{c}(\mathfrak{p}), 4D^{\mathrm{s}(\mathfrak{p})})$. Since

$$\mathtt{c}(\mathfrak{p}) \in \mathcal{I}(\mathfrak{p}) \subset B(c(J'), 100D^{s(J')+1}),$$

we have by `eq-vol-sp-cube` and the triangle inequality

$$J \subset J' \subset B(c(J'), 4D^{s(J')}) \subset B(\mathtt{c}(\mathfrak{p}), 104D^{s(J')+1}).$$

In particular this implies $104D^{s(J')+1} > 4D^{\mathrm{s}(\mathfrak{p})}$. By the triangle inequality we also have

$$B(\mathtt{c}(\mathfrak{p}), 104D^{s(J')+1}) \subset B(c(J), 204D^{s(J')+1}),$$

so from `measure-comparison`,

$$\mu(B(\mathtt{c}(\mathfrak{p}), 104D^{s(J')+1})) \leq 2^{200a^3+10a}\mu(J),$$

which proves $\mathfrak{p}$ satisfies the needed criteria with $r = 104D^{s(J')+1}$.

Almost orthogonality of separated trees

The main result of this subsection is the almost orthogonality estimate for operators associated to distinct trees in a forest in Lemma 1.7.4.4 below. We will deduce it from Lemmas Lemma 1.7.4.5 and Lemma 1.7.4.6, which are proven in Subsections section 1.7 (p. 95) and section 1.7 (p. 107), respectively. Before stating it, we introduce some relevant notation.

The adjoint of the operator $T_{\mathfrak{p}}$ defined in `definetp` is given by

$$T_{\mathfrak{p}}^* g(x) = \int_{E(\mathfrak{p})} \overline{K_{\mathsf{s}(\mathfrak{p})}(y,x)} e(-Q(y)(x) + Q(y)(y)) g(y) \, \mathrm{d}\mu(y).$$

Lemma 1.7.4.1

For each $\mathfrak{p} \in \mathfrak{P}$, we have

$$T_{\mathfrak{p}}^* g = \mathbf{1}_{B(\mathsf{c}(\mathfrak{p}), 5D^{\mathsf{s}(\mathfrak{p})})} T_{\mathfrak{p}}^* \mathbf{1}_{\mathcal{I}(\mathfrak{p})} g.$$

For each $\mathfrak{u} \in \mathfrak{U}$ and each $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$, we have

$$T_{\mathfrak{p}}^* g = \mathbf{1}_{\mathcal{I}(\mathfrak{u})} T_{\mathfrak{p}}^* \mathbf{1}_{\mathcal{I}(\mathfrak{u})} g.$$

Proof for Lemma 1.7.4.1

Proof. By `forest1`, $E(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{u})$. Thus by `definetp*`

$$T_{\mathfrak{p}}^* g(x) = T_{\mathfrak{p}}^*(\mathbf{1}_{\mathcal{I}(\mathfrak{p})} g)(x)$$

$$= \int_{E(\mathfrak{p})} \overline{K_{\mathsf{s}(\mathfrak{p})}(y,x)} e(-Q(y)(x) + Q(y)(y)) \mathbf{1}_{\mathcal{I}(\mathfrak{p})}(y) g(y) \, \mathrm{d}\mu(y).$$

If this integral is not 0, then there exists $y \in \mathcal{I}(\mathfrak{p})$ such that $K_{\mathsf{s}(\mathfrak{p})}(y,x) \neq 0$. By `supp-Ks`, `eq-vol-sp-cube` and the triangle inequality, it follows that

$$x \in B(\mathsf{c}(\mathfrak{p}), 5D^{\mathsf{s}(\mathfrak{p})}).$$

Thus

$$T_{\mathfrak{p}}^* g(x) = \mathbf{1}_{B(\mathsf{c}(\mathfrak{p}), 5D^{\mathsf{s}(\mathfrak{p})})}(x) T_{\mathfrak{p}}^*(\mathbf{1}_{\mathcal{I}(\mathfrak{p})} g)(x).$$

The second claimed equation follows now since $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{u})$ and by `forest6` we have $B(\mathsf{c}(\mathfrak{p}), 5D^{\mathsf{s}(\mathfrak{p})}) \subset \mathcal{I}(\mathfrak{u})$.

Lemma 1.7.4.2

For all bounded g supported on G we have that

$$\left\| \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}}^* g \right\|_2 \le 2^{181a^3} \operatorname{dens}_1(\mathfrak{T}(\mathfrak{u}))^{1/2} \|g\|_2$$

$$\left\| \mathbf{1}_F \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}}^* g \right\|_2 \le 2^{282a^3} \operatorname{dens}_1(\mathfrak{T}(\mathfrak{u}))^{1/2} \operatorname{dens}_2(\mathfrak{T}(\mathfrak{u}))^{1/2} \|g\|_2.$$

Proof for Lemma 1.7.4.2

Proof. By Lemma 1.7.3.1, we have for all bounded f and g with $|g| \leq \mathbf{1}_G$ that

$$\left|\int_X \overline{\sum_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} T^*_{\mathfrak{p}} g} f \,\mathrm{d}\mu\right| = \left|\int_X \overline{g} \sum_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f \,\mathrm{d}\mu\right|$$
$$\leq 2^{181a^3} \operatorname{dens}_1(\mathfrak{T}(\mathfrak{u}))^{1/2} \|g\|_2 \|f\|_2.$$

Let $f = \sum_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} T^*_{\mathfrak{p}} g$. Since $|g| \leq \mathbf{1}_G$, f is bounded and has bounded support. In particular $\|f\|_2 < \infty$. Dividing `eq-adjoint-bound` by $\|f\|_2$ completes the proof.The proof of the second part is similar with $f = \mathbf{1}_F \sum_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} T^*_{\mathfrak{p}} g$.

We define

$$S_{2,\mathfrak{u}} g := \left|\sum_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} T^*_{\mathfrak{p}} g\right| + M_{\mathcal{B},1} g + |g|.$$

Lemma 1.7.4.3

We have for all $\mathfrak{u} \in \mathfrak{U}$ and all bounded g supported on G

$$\|S_{2,\mathfrak{u}} g\|_2 \leq 2^{182a^3} \|g\|_2.$$

Proof for Lemma 1.7.4.3

Proof. This follows immediately from Minkowski's inequality, Theorem 1.2.6 and Lemma 1.7.4.2, using that $a \geq 4$.

Now we are ready to state the main result of this subsection.

Lemma 1.7.4.4

For any $\mathfrak{u}_1 \neq \mathfrak{u}_2 \in \mathfrak{U}$ and all bounded g_1, g_2 with bounded support, we have

$$\left|\int_X \sum_{\mathfrak{p}_1\in\mathfrak{T}(\mathfrak{u}_1)} \sum_{\mathfrak{p}_2\in\mathfrak{T}(\mathfrak{u}_2)} T^*_{\mathfrak{p}_1} g_1 \overline{T^*_{\mathfrak{p}_2} g_2} \,\mathrm{d}\mu\right|$$
$$\leq 2^{512a^3-4n} \prod_{j=1}^{2} \|S_{2,\mathfrak{u}_j} g_j\|_{L^2(\mathcal{I}(\mathfrak{u}_1)\cap\mathcal{I}(\mathfrak{u}_2))}.$$

Proof for Lemma 1.7.4.4

Proof of Lemma 1.7.4.4. By Lemma 1.7.4.1 and `dyadicproperty`, the left hand side `eq-lhs-sep-tree` is 0 unless $\mathcal{I}(\mathfrak{u}_1) \subset \mathcal{I}(\mathfrak{u}_2)$ or $\mathcal{I}(\mathfrak{u}_2) \subset \mathcal{I}(\mathfrak{u}_1)$. Without loss of generality we assume that $\mathcal{I}(\mathfrak{u}_1) \subset \mathcal{I}(\mathfrak{u}_2)$.Define

$$\mathfrak{S} := \{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_1) \cup \mathfrak{T}(\mathfrak{u}_2) \ : \ d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)) \geq 2^{Zn/2}\}.$$

Lemma 1.7.4.4 follows by combining the definition `defineZ` of Z with the following two lemmas.

Lemma 1.7.4.5

We have for all $\mathfrak{u}_1 \neq \mathfrak{u}_2 \in \mathfrak{U}$ with $\mathcal{I}(\mathfrak{u}_1) \subset \mathcal{I}(\mathfrak{u}_2)$ and all bounded g_1, g_2 with bounded support

$$\left| \int_X \sum_{\mathfrak{p}_1 \in \mathfrak{T}(\mathfrak{u}_1)} \sum_{\mathfrak{p}_2 \in \mathfrak{T}(\mathfrak{u}_2) \cap \mathfrak{S}} T^*_{\mathfrak{p}_1} g_1 \overline{T^*_{\mathfrak{p}_2} g_2} \, \mathrm{d}\mu \right|$$

$$\leq 2^{511 a^3} 2^{-Zn/(4a^2+2a^3)} \prod_{j=1}^{2} \|S_{2,\mathfrak{u}_j} g_j\|_{L^2(\mathcal{I}(\mathfrak{u}_1))}.$$

Lemma 1.7.4.6

We have for all $\mathfrak{u}_1 \neq \mathfrak{u}_2 \in \mathfrak{U}$ with $\mathcal{I}(\mathfrak{u}_1) \subset \mathcal{I}(\mathfrak{u}_2)$ and all bounded g_1, g_2 with bounded support

$$\left| \int_X \sum_{\mathfrak{p}_1 \in \mathfrak{T}(\mathfrak{u}_1)} \sum_{\mathfrak{p}_2 \in \mathfrak{T}(\mathfrak{u}_2) \setminus \mathfrak{S}} T^*_{\mathfrak{p}_1} g_1 \overline{T^*_{\mathfrak{p}_2} g_2} \, \mathrm{d}\mu \right|$$

$$\leq 2^{232 a^3 + 21 a + 5} 2^{-\frac{25}{101 a} Zn\kappa} \prod_{j=1}^{2} \|S_{2,\mathfrak{u}_j} g_j\|_{L^2(\mathcal{I}(\mathfrak{u}_1))}.$$

In the proofs of both lemmas, we will need the following observation.

Lemma 1.7.4.7

Let $\mathfrak{u}_1 \neq \mathfrak{u}_2 \in \mathfrak{U}$ with $\mathcal{I}(\mathfrak{u}_1) \subset \mathcal{I}(\mathfrak{u}_2)$. If $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_1) \cup \mathfrak{T}(\mathfrak{u}_2)$ with $\mathcal{I}(\mathfrak{p}) \cap \mathcal{I}(\mathfrak{u}_1) \neq \emptyset$, then $\mathfrak{p} \in \mathfrak{S}$. In particular, we have $\mathfrak{T}(\mathfrak{u}_1) \subset \mathfrak{S}$.

Proof for Lemma 1.7.4.7

Proof. Suppose first that $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_1)$. Then $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{u}_1) \subset \mathcal{I}(\mathfrak{u}_2)$, by `forest1`. Thus we have by the separation condition `forest5`, `eq-freq-comp-ball`, `forest1` and the triangle inequality

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)) \geq d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{u}_2)) - d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{u}_1))$$

$$\geq 2^{Z(n+1)} - 4$$
$$\geq 2^{Zn/2},$$

using that $Z = 2^{12a} \geq 4$. Hence $\mathfrak{p} \in \mathfrak{S}$.Suppose now that $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_2)$. If $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{u}_1)$, then the same argument as above with $\mathfrak{u}_1$ and $\mathfrak{u}_2$ swapped shows $\mathfrak{p} \in \mathfrak{S}$. If $\mathcal{I}(\mathfrak{p}) \not\subset \mathcal{I}(\mathfrak{u}_1)$ then, by `dyadicproperty`, $\mathcal{I}(\mathfrak{u}_1) \subset \mathcal{I}(\mathfrak{p})$. Pick $\mathfrak{p}' \in \mathfrak{T}(\mathfrak{u}_1)$, we have $\mathcal{I}(\mathfrak{p}') \subset \mathcal{I}(\mathfrak{u}_1) \subset \mathcal{I}(\mathfrak{p})$. Hence, by Lemma 1.2.1.2 and the first paragraph

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)) \geq d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)) \geq 2^{Zn},$$

so $\mathfrak{p} \in \mathfrak{S}$.

To simplify the notation, we will write at various places throughout the proof of Lemmas Lemma 1.7.4.5 and Lemma 1.7.4.6 for a subset $\mathfrak{C} \subset \mathfrak{P}$

$$T_{\mathfrak{C}} f := \sum_{\mathfrak{p} \in \mathfrak{C}} T_{\mathfrak{p}} f \,, \qquad T_{\mathfrak{C}}^* g := \sum_{\mathfrak{p} \in \mathfrak{C}} T_{\mathfrak{p}}^* g \,.$$

Proof of the Tiles with large separation Lemma

Lemma 1.7.4.5 follows from the van der Corput estimate in Theorem 1.2.5. We apply this proposition in section 1.7 (p. 105). To prepare this application, we first, in section 1.7 (p. 95), construct a suitable partition of unity, and show then, in section 1.7 (p. 98) the Holder estimates needed to apply Theorem 1.2.5.

A partition of unity

Define

$$\mathcal{J}' = \{J \in \mathcal{J}(\mathfrak{S}) \ : \ J \subset \mathcal{I}(\mathfrak{u}_1)\}.$$

Lemma 1.7.5.1.1

We have that

$$\mathcal{I}(\mathfrak{u}_1) = \dot{\bigcup_{J \in \mathcal{J}'}} J.$$

Proof for Lemma 1.7.5.1.1

Proof. By Lemma 1.7.1.2, it remains only to show that each $J \in \mathcal{J}(\mathfrak{S})$ with $J \cap \mathcal{I}(\mathfrak{u}_1) \neq \emptyset$ is in $\mathcal{J}'$. But if $J \notin \mathcal{J}'$, then by `dyadicproperty` $\mathcal{I}(\mathfrak{u}_1) \subset J$. Pick $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_1) \subset \mathfrak{S}$. Then $\mathcal{I}(\mathfrak{p}) \subset J$. This contradicts the definition of $\mathcal{J}(\mathfrak{S})$.

For cubes $J \in \mathcal{D}$, denote

$$B(J) := B(c(J), 8D^{s(J)}).$$

The main result of this subsubsection is the following.
Lemma 1.7.5.1.2

There exists a family of functions χ_J, $J \in \mathcal{J}'$ such that

$$\mathbf{1}_{\mathcal{I}(\mathfrak{u}_1)} = \sum_{J \in \mathcal{J}'} \chi_J,$$

and for all $J \in \mathcal{J}'$ and all $y, y' \in \mathcal{I}(\mathfrak{u}_1)$

$$0 \le \chi_J(y) \le \mathbf{1}_{B(J)}(y),$$

$$|\chi_J(y) - \chi_J(y')| \le 2^{227a^3} \frac{\rho(y, y')}{D^{s(J)}}.$$

In the proof, we will use the following auxiliary lemma.

Lemma 1.7.5.1.3

If $J, J' \in \mathcal{J}'$ with

$$B(J) \cap B(J') \ne \emptyset,$$

then $|s(J) - s(J')| \le 1$.

Proof for Lemma 1.7.5.1.2

Proof of Lemma 1.7.5.1.2. For each cube $J \in \mathcal{J}$ let

$$\tilde{\chi}_J(y) = \mathbf{1}_{\mathcal{I}(\mathfrak{u}_1)}(y) \max\{0, 8 - D^{-s(J)}\rho(y, c(J))\},$$

and set

$$a(y) = \sum_{J \in \mathcal{J}'} \tilde{\chi}_J(y).$$

We define

$$\chi_J(y) := \frac{\tilde{\chi}_J(y)}{a(y)}.$$

Then, due to `forest6` and `def-BJ`, the properties `eq-pao-1` and `eq-pao-2` are clearly true. Estimate `eq-pao-3` follows from `eq-pao-2` if $y, y' \notin B(J)$. Thus we can assume that $y \in B(J)$. We have by the triangle inequality

$$|\chi_J(y) - \chi_J(y')| \le \frac{|\tilde{\chi}_J(y) - \tilde{\chi}_J(y')|}{a(y)} + \frac{\tilde{\chi}_J(y')|a(y) - a(y')|}{a(y)a(y')}$$

Since $\tilde{\chi}_J(z) \ge 4$ for all $z \in B(c(J), 4D^{s(J)}) \supset J$ and by Lemma 1.7.5.1.1, we have that $a(z) \ge 4$ for all $z \in \mathcal{I}(\mathfrak{u}_1)$. So we can estimate the above further by

$$\le 2^{-2}(|\tilde{\chi}_J(y) - \tilde{\chi}_J(y')| + \tilde{\chi}_J(y')|a(y) - a(y')|).$$

If $y' \notin B(\mathrm{c}(\mathfrak{p}), 8D^{\mathrm{s}(\mathfrak{p})})$ then the second summand vanishes. Else, we can estimate the above, using also that $|\tilde{\chi}_J(y')| \le 8$, by

$$\le 2^{-2}|\tilde{\chi}_J(y) - \tilde{\chi}_J(y')| + 2 \sum_{\substack{J' \in \mathcal{J}' \\ B(J') \cap B(J) \ne \emptyset}} |\tilde{\chi}_{J'}(y) - \tilde{\chi}_{J'}(y')|.$$

By the triangle inequality, we have for all dyadic cubes I

$$|\tilde{\chi}_I(y) - \tilde{\chi}_I(y')| \le \rho(y, y') D^{-s(I)}.$$

Using this above, we obtain

$$|\chi_J(y) - \chi_J(y')| \le \rho(y, y')\Big(\frac{1}{4} D^{-s(J)} + 2 \sum_{\substack{J' \in \mathcal{J}' \\ B(J') \cap B(J) \ne \emptyset}} D^{-s(J')}\Big).$$

By Lemma 1.7.5.1.3, this is at most

$$\frac{\rho(y, y')}{D^{s(J)}} \left(\frac{1}{4} + 2D|\{J' \in \mathcal{J}' \ : \ B(J') \cap B(J) \ne \emptyset\}|\right).$$

By `eq-vol-sp-cube` and Lemma 1.7.5.1.1, the balls $B(c(J'), \frac{1}{4}D^{s(J')})$ are pairwise disjoint. By the triangle inequality and Lemma 1.7.5.1.3, each such ball for J' in the set of the last display is contained in

$$B(c(J), 9D^{s(J)+1}).$$

By the doubling property `doublingx`, we further have

$$\mu(B(c(J), 9D^{s(J)+1})) \le 2^{200a^3+7a} \mu\Big(B(c(J'), \frac{1}{4}D^{s(J')})\Big)$$

for each such ball. Thus

$$|\{J' \in \mathcal{J}' \ : \ B(J') \cap B(J) \ne \emptyset\}| \le 2^{200a^3+7a}.$$

Recalling that $D = 2^{100a^2}$, we obtain

$$\frac{1}{4} + 2D|\{J' \in \mathcal{J}' \ : \ B(J') \cap B(J) \ne \emptyset\}| \le 2^{200a^3+100a^2+7a+2}.$$

Since $a \ge 4$, `eq-pao-3` follows.

Proof for Lemma 1.7.5.1.3

Proof of Lemma 1.7.5.1.3. Suppose that $s(J') < s(J) - 1$. Then $s(J) > -S$. Thus, by the definition of $\mathcal{J}'$ there exists no $\mathfrak{p} \in \mathfrak{S}$ with

$$\mathcal{I}(\mathfrak{p}) \subset B(c(J), 100D^{s(J)+1}).$$

Since $s(J') < s(J)$, there exists a cube $J'' \in \mathcal{D}$ with $J \subset J''$ and $s(J'') = s(J') + 1$. By the definition of $\mathcal{J}'$, there exists a tile $\mathfrak{p} \in \mathfrak{S}$ with

$$\mathcal{I}(\mathfrak{p}) \subset B(c(J''), 100D^{s(J')+2}).$$

But by the triangle inequality and `defineD`, we have

$$B(c(J''), 100D^{s(J')+2}) \subset B(c(J), 100D^{s(J)+1}),$$

which contradicts `eq-tile-incl-1` and `tile-incl-2`.

Holder estimates for adjoint tree operators

Let $g_1, g_2 : X \to \mathbb{C}$ be bounded with bounded support. Define for $J \in \mathcal{J}'$

$$h_J(y) := \chi_J(y) \cdot (e(\mathcal{Q}(\mathfrak{u}_1)(y)) T^*_{\mathfrak{T}(\mathfrak{u}_1)} g_1(y)) \cdot \overline{(e(\mathcal{Q}(\mathfrak{u}_2)(y)) T^*_{\mathfrak{T}(\mathfrak{u}_2)\cap\mathfrak{S}} g_2(y))}.$$

The main result of this subsubsection is the following τ-Holder estimate for h_J, where $\tau = 1/a$.

Lemma 1.7.5.2.1

We have for all $J \in \mathcal{J}'$ that

$$\|h_J\|_{C^\tau(B(c(J),16D^{s(J)}))} \le 2^{485a^3} \prod_{j=1,2} (\inf_{B(c(J),\frac{1}{8}D^{s(J)})} |T^*_{\mathfrak{T}(\mathfrak{u}_j)} g_j| + \inf_J M_{\mathcal{B},1}|g_j|).$$

We will prove this lemma at the end of this section, after establishing several auxiliary results.

We begin with the following Holder continuity estimate for adjoints of operators associated to tiles.

Lemma 1.7.5.2.2

Let $\mathfrak{u} \in \mathfrak{U}$ and $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$. Then for all $y, y' \in X$ and all bounded g with bounded support, we have

$$|e(\mathcal{Q}(\mathfrak{u})(y)) T^*_{\mathfrak{p}} g(y) - e(\mathcal{Q}(\mathfrak{u})(y')) T^*_{\mathfrak{p}} g(y')|$$
$$\le \frac{2^{128a^3}}{\mu(B(\mathrm{c}(\mathfrak{p}), 4D^{\mathrm{s}(\mathfrak{p})}))} \left(\frac{\rho(y,y')}{D^{\mathrm{s}(\mathfrak{p})}}\right)^{1/a} \int_{E(\mathfrak{p})} |g(x)| \, \mathrm{d}\mu(x).$$

Proof for Lemma 1.7.5.2.2

Proof. By `definetp*`, we have

$$|e(\mathcal{Q}(\mathfrak{u})(y)) T^*_{\mathfrak{p}} g(y) - e(\mathcal{Q}(\mathfrak{u})(y')) T^*_{\mathfrak{p}} g(y')|$$
$$= \Big| \int_{E(\mathfrak{p})} e(Q(x)(x) - Q(x)(y) + \mathcal{Q}(\mathfrak{u})(y)) \overline{K_{\mathrm{s}(\mathfrak{p})}(x,y)} g(x) - e(Q(x)(x) - Q(x)(y') + \mathcal{Q}(\mathfrak{u})(y')) \overline{K_{\mathrm{s}(\mathfrak{p})}(x,y')} g(x) \, \mathrm{d}$$
$$\le \int_{E(\mathfrak{p})} |g(x)| |e(Q(x)(y) - Q(x)(y') - \mathcal{Q}(\mathfrak{u})(y) + \mathcal{Q}(\mathfrak{u})(y')) \overline{K_{\mathrm{s}(\mathfrak{p})}(x,y)} - \overline{K_{\mathrm{s}(\mathfrak{p})}(x,y')}| \, \mathrm{d}\mu(x)$$
$$\le \int_{E(\mathfrak{p})} |g(x)| |e(-Q(x)(y) + Q(x)(y') + \mathcal{Q}(\mathfrak{u})(y) - \mathcal{Q}(\mathfrak{u})(y')) - 1| \times |\overline{K_{\mathrm{s}(\mathfrak{p})}(x,y)}| \, \mathrm{d}\mu(x)$$
$$+ \int_{E(\mathfrak{p})} |g(x)| |\overline{K_{\mathrm{s}(\mathfrak{p})}(x,y)} - \overline{K_{\mathrm{s}(\mathfrak{p})}(x,y')}| \, \mathrm{d}\mu(x).$$

By the oscillation estimate `osccontrol`, we have

$$| - Q(x)(y) + Q(x)(y') + \mathcal{Q}(\mathfrak{u})(y) - \mathcal{Q}(\mathfrak{u})(y')|$$

$$\leq d_{B(y,1.6\rho(y,y'))}(Q(x), \mathcal{Q}(\mathfrak{u})).$$

Suppose that $y, y' \in B(\mathsf{c}(\mathfrak{p}), 5D^{\mathsf{s}(\mathfrak{p})})$, so that $\rho(y,y') \leq 10D^{\mathsf{s}(\mathfrak{p})}$. Let $k \in \mathbb{Z}$ be such that $2^{ak}\rho(y,y') \leq 10D^{\mathsf{s}(\mathfrak{p})}$ but $2^{a(k+1)}\rho(y,y') > 10D^{\mathsf{s}(\mathfrak{p})}$. In particular, $k \geq 0$. Then, using `seconddb` followed by `firstdb`, we can bound `eq-lem-tile-Holder-comp` from above by

$$2^{-k} d_{B(\mathsf{c}(\mathfrak{p}),16D^{\mathsf{s}(\mathfrak{p})})}(Q(x), \mathcal{Q}(\mathfrak{u})) \leq 2^{6a-k} d_{\mathfrak{p}}(Q(x), \mathcal{Q}(\mathfrak{u})).$$

Since $x \in E(\mathfrak{p})$ we have $Q(x) \in \Omega(\mathfrak{p}) \subset B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 1)$, and since $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ we have $\mathcal{Q}(\mathfrak{u}) \in B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 4)$, so this is estimated by

$$\leq 5 \cdot 2^{6a-k}.$$

By definition of k, we have

$$-k < 1 - \frac{1}{a}\log_2\left(\frac{10D^{\mathsf{s}(\mathfrak{p})}}{\rho(y,y')}\right),$$

which gives

$$|-Q(x)(y)+Q(x)(y')+\mathcal{Q}(\mathfrak{u})(y)-\mathcal{Q}(\mathfrak{u})(y')| \leq 10\cdot 2^{6a}\left(\frac{\rho(y,y')}{10D^{\mathsf{s}(\mathfrak{p})}}\right)^{1/a}.$$

For all $x \in \mathcal{I}(\mathfrak{p})$, we have by `doublingx` that

$$\mu(B(x, D^{\mathsf{s}(\mathfrak{p})})) \geq 2^{-3a}\mu(B(\mathsf{c}(\mathfrak{p}), 4D^{\mathsf{s}(\mathfrak{p})})).$$

Combining the above with `eq-Ks-size`, `eq-Ks-smooth` and `eq-lem-Tile-holder-im1`, we obtain that the sum of the terms in `T*Holder1b` and `T*Holder1` is bounded by

$$\frac{2^{3a}}{\mu(B(\mathsf{c}(\mathfrak{p}), 4D^{\mathsf{s}(\mathfrak{p})}))}\int_{E(\mathfrak{p})} |g(x)| \, \mathrm{d}\mu(x) \times (2^{102a^3}\cdot 10\cdot 2^{6a}\left(\frac{\rho(y,y')}{D^{\mathsf{s}(\mathfrak{p})}}\right)^{1/a} + 2^{127a^3}\left(\frac{\rho(y,y')}{D^{\mathsf{s}(\mathfrak{p})}}\right)^{1/a})$$

Since $\rho(y,y') \leq 10D^{\mathsf{s}(\mathfrak{p})}$, we conclude that the sum of the terms in `T*Holder1b` and `T*Holder1` is bounded by

$$\frac{2^{128a^3}}{\mu(B(\mathsf{c}(\mathfrak{p}), 4D^{\mathsf{s}(\mathfrak{p})}))}\left(\frac{\rho(y,y')}{D^{\mathsf{s}(\mathfrak{p})}}\right)^{1/a}\int_{E(\mathfrak{p})} |g(x)| \, \mathrm{d}\mu(x).$$

Next, if $y, y' \notin B(\mathsf{c}(\mathfrak{p}), 5D^{\mathsf{s}(\mathfrak{p})})$, then $T^*_{\mathfrak{p}}g(y) = T^*_{\mathfrak{p}}g(y') = 0$, by Lemma 1.7.4.1. Then `T*Holder2` holds.Finally, if $y \in B(\mathsf{c}(\mathfrak{p}), 5D^{\mathsf{s}(\mathfrak{p})})$ and $y' \notin B(\mathsf{c}(\mathfrak{p}), 5D^{\mathsf{s}(\mathfrak{p})})$, then

$$|e(\mathcal{Q}(\mathfrak{u})(y))T^*_{\mathfrak{p}}g(y) - e(\mathcal{Q}(\mathfrak{u})(y'))T^*_{\mathfrak{p}}g(y')| = |T^*_{\mathfrak{p}}g(y)|$$

$$\leq \int_{E(\mathfrak{p})} |K_{\mathsf{s}(\mathfrak{p})}(x,y)||g(x)|\,\mathrm{d}\mu(x).$$

By the same argument used to prove `eq-Ks-aux`, this is bounded by

$$\leq 2^{102a^3} \int_{E(\mathfrak{p})} \frac{1}{\mu(B(x,D^s))} \psi(D^{-s}\rho(x,y))|g(x)|\,\mathrm{d}\mu(x).$$

It follows from the definition of ψ that

$$\psi(x) \leq \max\{0,(2-4x)^{1/a}\}.$$

Now for all $x \in E(\mathfrak{p})$, it follows by the triangle inequality and `eq-vol-sp-cube` that

$$2-4D^{-\mathsf{s}(\mathfrak{p})}\rho(x,y) \leq 2-4D^{-\mathsf{s}(\mathfrak{p})}\rho(y,\mathsf{c}(\mathfrak{p}))+4D^{-\mathsf{s}(\mathfrak{p})}\rho(x,\mathsf{c}(\mathfrak{p})) \leq 18-4D^{-\mathsf{s}(\mathfrak{p})}\rho(y,\mathsf{c}(\mathfrak{p})) \leq 4D^{-\mathsf{s}(\mathfrak{p})}\rho(y,y')-2$$

Combining the above with the previous estimate on ψ, we get

$$\psi(D^{-\mathsf{s}(\mathfrak{p})}\rho(x,y)) \leq 4(D^{-\mathsf{s}(\mathfrak{p})}\rho(y,y'))^{1/a}.$$

Further, we obtain from the doubling property `doublingx` and `eq-vol-sp-cube` that

$$\mu(B(x,D^{\mathsf{s}(\mathfrak{p})})) \geq 2^{-3a}\mu(B(\mathsf{c}(\mathfrak{p}),4D^{\mathsf{s}(\mathfrak{p})})).$$

Plugging this into `eq-lem-Tile-holder-im2` and using $a \geq 4$, we get

$$|T_{\mathfrak{p}}^* g(y)| \leq \frac{2^{103a^3}}{\mu(B(\mathsf{c}(\mathfrak{p}),4D^{\mathsf{s}(\mathfrak{p})}))} \left(\frac{\rho(y,y')}{D^{\mathsf{s}(\mathfrak{p})}}\right)^{1/a} \int_{E(\mathfrak{p})} |g(x)|\,\mathrm{d}\mu(y),$$

which completes the proof of the lemma.

Recall that

$$B(J) := B(c(J),8D^{s(J)}).$$

We also denote

$$B'(J) := B(c(J),16D^{s(J)}),$$

$$B^\circ(J) := B(c(J),\frac{1}{8}D^{s(J)}).$$

Lemma 1.7.5.2.3

Let $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_2) \setminus \mathfrak{S}$, $J \in \mathcal{J}'$ and suppose that

$$B(\mathcal{I}(\mathfrak{p})) \cap B^\circ(J) \neq \emptyset.$$

Then

$$s(J) \leq \mathsf{s}(\mathfrak{p}) \leq s(J)+3.$$

Proof for Lemma 1.7.5.2.3

Proof. For the first estimate, assume that $\mathrm{s}(\mathfrak{p}) < s(J)$, then in particular $\mathrm{s}(\mathfrak{p}) \le \mathrm{s}(\mathfrak{u}_1)$. Since $\mathfrak{p} \notin \mathfrak{S}$, we have by Lemma 1.7.4.7 that $\mathcal{I}(\mathfrak{p}) \cap \mathcal{I}(\mathfrak{u}_1) = \emptyset$. Since $B\Big(c(J), \frac{1}{4}D^{s(J)}\Big) \subset \mathcal{I}(J) \subset \mathcal{I}(\mathfrak{u}_1)$, this implies

$$\rho(c(J), \mathtt{c}(\mathfrak{p})) \ge \frac{1}{4}D^{s(J)}.$$

On the other hand

$$\rho(c(J), \mathtt{c}(\mathfrak{p})) \le \frac{1}{8}D^{s(J)} + 8D^{\mathrm{s}(\mathfrak{p})},$$

by our assumption. Thus $D^{\mathrm{s}(\mathfrak{p})} \ge 64^{-1}D^{s(J)}$, which contradicts `defineD` and $a \ge 4$.For the second estimate, assume that $\mathrm{s}(\mathfrak{p}) > s(J) + 3$. Since $J \in \mathcal{J}'$, we have $J \subsetneq \mathcal{I}(\mathfrak{u}_1)$. Thus there exists $J' \in \mathcal{D}$ with $J \subset J'$ and $s(J') = s(J) + 1$, by `coverdyadic` and `dyadicproperty`. By definition of $\mathcal{J}'$, there exists some $\mathfrak{p}' \in \mathfrak{S}$ such that $\mathcal{I}(\mathfrak{p}') \subset B(c(J'), 100D^{s(J)+2})$. On the other hand, since $B(\mathcal{I}(\mathfrak{p})) \cap B^\circ(J) \neq \emptyset$, by the triangle inequality it holds that

$$B(c(J'), 100D^{s(J)+3}) \subset B(\mathtt{c}(\mathfrak{p}), 10D^{\mathrm{s}(\mathfrak{p})}).$$

Using the definition of $\mathfrak{S}$, we have

$$2^{Zn/2} \le d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)) \le d_{B(c(J'),100D^{s(J)+2})}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)).$$

By `seconddb`, this is

$$\le 2^{-100a} d_{B(c(J'),100D^{s(J)+3})}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2))$$

$$\le 2^{-100a} d_{B(\mathtt{c}(\mathfrak{p}),10D^{\mathrm{s}(\mathfrak{p})})}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)),$$

and by `firstdb` and the definition of $\mathfrak{S}$

$$\le 2^{-94a} d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)) \le 2^{-94a}2^{Zn/2}.$$

This is a contradiction, the second estimate follows.

Lemma 1.7.5.2.4

For all $J \in \mathcal{J}'$ and all bounded g with bounded support

$$\sup_{B^\circ(J)} |T^*_{\mathfrak{T}(\mathfrak{u}_2)\setminus\mathfrak{S}}g| \le 2^{104a^3} \inf_J M_{\mathcal{B},1}|g|$$

Proof for Lemma 1.7.5.2.4

Proof. By the triangle inequality and since $T_{\mathfrak{p}}^* g = \mathbf{1}_{B(\mathrm{c}(\mathfrak{p}), 5D^{\mathrm{s}(\mathfrak{p})})} T_{\mathfrak{p}}^* g$, we have

$$\sup_{B^\circ(J)} |T^*_{\mathfrak{T}(\mathfrak{u}_2)\setminus\mathfrak{S}} g| \le \sup_{B^\circ(J)} \sum_{\substack{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u}_2)\setminus\mathfrak{S} \\ B(\mathcal{I}(\mathfrak{p}))\cap B^\circ(J)\neq\emptyset}} |T_{\mathfrak{p}}^* g|.$$

By Lemma 1.7.5.2.3, this is at most

$$\sum_{s=s(J)}^{s(J)+3} \sum_{\substack{\mathfrak{p}\in\mathfrak{P}, \mathrm{s}(\mathfrak{p})=s \\ B(\mathcal{I}(\mathfrak{p}))\cap B^\circ(J)\neq\emptyset}} \sup_{B^\circ(J)} |T_{\mathfrak{p}}^* g|.$$

If $x \in E(\mathfrak{p})$ and $B(\mathcal{I}(\mathfrak{p})) \cap B^\circ(J) \neq \emptyset$, then

$$B(c(J), 16D^{\mathrm{s}(\mathfrak{p})}) \subset B(x, 32D^{\mathrm{s}(\mathfrak{p})}),$$

by `eq-vol-sp-cube` and the triangle inequality. Using the doubling property `doublingx`, it follows that

$$\mu(B(x, D^{\mathrm{s}(\mathfrak{p})})) \ge 2^{-5a} \mu(B(c(J), 16D^{\mathrm{s}(\mathfrak{p})})).$$

Using `definetp*`, `eq-Ks-size` and that $a \ge 4$, we bound `eq-sep-tree-aux-3` by

$$2^{103a^3} \sum_{s=s(J)}^{s(J)+3} \sum_{\substack{\mathfrak{p}\in\mathfrak{P}, \mathrm{s}(\mathfrak{p})=s \\ B(\mathcal{I}(\mathfrak{p}))\cap B^\circ(J)\neq\emptyset}} \frac{1}{\mu(B(c(J), 16D^s)} \int_{E(\mathfrak{p})} |g| \, \mathrm{d}\mu.$$

For each $I \in \mathcal{D}$, the sets $E(\mathfrak{p})$ for $\mathfrak{p} \in \mathfrak{P}$ with $\mathcal{I}(\mathfrak{p}) = I$ are pairwise disjoint by `defineep` and `eq-dis-freq-cover`. Further, if $B(\mathcal{I}(\mathfrak{p}))\cap B^\circ(J) \neq \emptyset$ and $\mathrm{s}(\mathfrak{p}) \ge s(J)$, then $E(\mathfrak{p}) \subset B(c(J), 16D^{\mathrm{s}(\mathfrak{p})})$. Thus the last display is bounded by

$$2^{103a^3} \sum_{s=s(J)}^{s(J)+3} \frac{1}{\mu(B(c(J), 16D^s))} \int_{B(c(J),16D^s)} |g| \, \mathrm{d}\mu.$$

$$\le \inf_{x'\in J} 2^{103a^3+2} M_{\mathcal{B},1}|g|.$$

The lemma follows since $a \ge 4$.

Lemma 1.7.5.2.5

Let $\mathfrak{C} = \mathfrak{T}(\mathfrak{u}_1)$ or $\mathfrak{C} = \mathfrak{T}(\mathfrak{u}_2) \cap \mathfrak{S}$. Then for each $J \in \mathcal{J}'$ and $\mathfrak{p} \in \mathfrak{C}$ with $B(\mathcal{I}(\mathfrak{p})) \cap B'(J) \neq \emptyset$, we have $\mathrm{s}(\mathfrak{p}) \ge s(J)$.

Proof for Lemma 1.7.5.2.5

Proof. By Lemma 1.7.4.7, we have that in both cases, $\mathfrak{C} \subset \mathfrak{S}$. If $\mathfrak{p} \in \mathfrak{C}$ with $B(\mathcal{I}(\mathfrak{p})) \cap B'(J) \neq \emptyset$ and $\mathrm{s}(\mathfrak{p}) < s(J)$, then $\mathcal{I}(\mathfrak{p}) \subset B(c(J), 100D^{s(J)+1})$. Since $\mathfrak{p} \in \mathfrak{S}$, it follows from the definition of $\mathcal{J}'$ that $s(J) = -S$, which contradicts $\mathrm{s}(\mathfrak{p}) < s(J)$.

Lemma 1.7.5.2.6

Let $\mathfrak{C}_1 = \mathfrak{T}(\mathfrak{u}_1)$ and $\mathfrak{C}_2 = \mathfrak{T}(\mathfrak{u}_2) \cap \mathfrak{S}$. Then for $i = 1, 2$ and each $J \in \mathcal{J}'$ and all bounded g with bounded support, we have

$$\sup_{B'(J)} |T^*_{\mathfrak{C}_i} g| \leq \inf_{B^\circ(J)} |T^*_{\mathfrak{C}_i} g| + 2^{128a^3+4a+3} \inf_J M_{\mathcal{B},1}|g|$$

and for all $y, y' \in B'(J)$

$$|e(\mathcal{Q}(\mathfrak{u}_i)(y))T^*_{\mathfrak{C}_i} g(y) - e(\mathcal{Q}(\mathfrak{u}_i)(y'))T^*_{\mathfrak{C}_i} g(y')|$$

$$\leq 2^{128a^3+4a+1} \left(\frac{\rho(y,y')}{D^{s(J)}}\right)^{1/a} \inf_J M_{\mathcal{B},1}|g|.$$

Proof for Lemma 1.7.5.2.6

Proof. Note that `TreeUB` follows from `TreeHolder`, since for $y' \in B^\circ(J)$, by the triangle inequality,

$$\left(\frac{\rho(y,y')}{D^{s(J)}}\right)^{1/a} \leq \left(16 + \frac{1}{8}\right)^{1/a} \leq 2^2.$$

By the triangle inequality, Lemma 1.7.4.1 and Lemma 1.7.5.2.2, we have for all $y, y' \in B'(J)$

$$|e(\mathcal{Q}(\mathfrak{u}_i)(y))T^*_{\mathfrak{C}_i} g(y) - e(\mathcal{Q}(\mathfrak{u}_i)(y'))T^*_{\mathfrak{C}_i} g(y')|$$

$$\leq \sum_{\substack{\mathfrak{p} \in \mathfrak{C}_i \\ B(\mathcal{I}(\mathfrak{p})) \cap B'(J) \neq \emptyset}} |e(\mathcal{Q}(\mathfrak{u}_i)(y))T^*_{\mathfrak{p}} g(y) - e(\mathcal{Q}(\mathfrak{u}_i)(y'))T^*_{\mathfrak{p}} g(y')|$$

$$\leq 2^{128a^3} \rho(y,y')^{1/a} \sum_{\substack{\mathfrak{p} \in \mathfrak{C}_i \\ B(\mathcal{I}(\mathfrak{p})) \cap B'(J) \neq \emptyset}} \frac{D^{-\mathrm{s}(\mathfrak{p})/a}}{\mu(B(\mathrm{c}(\mathfrak{p}), 4D^{\mathrm{s}(\mathfrak{p})}))} \int_{E(\mathfrak{p})} |g| \, \mathrm{d}\mu.$$

By Lemma 1.7.5.2.5, we have $\mathrm{s}(\mathfrak{p}) \geq s(J)$ for all $\mathfrak{p}$ occurring in the sum. Further, for each $s \geq s(J)$, the sets $E(\mathfrak{p})$ for $\mathfrak{p} \in \mathfrak{P}$ with $\mathrm{s}(\mathfrak{p}) = s$ are pairwise disjoint by `defineep` and `eq-dis-freq-cover`, and contained in $B(c(J), 32D^s)$ by `eq-vol-sp-cube` and the triangle inequality. Using also the doubling estimate `doublingx`, we obtain that the expression in the last display can be estimated by

$$2^{128a^3} \rho(y,y')^{1/a} \sum_{S \geq s \geq s(J)} D^{-s/a} \frac{2^{4a}}{\mu(B(c(J), 32D^s))} \int_{B(c(J),32D^s)} |g| \, \mathrm{d}\mu$$

$$\leq 2^{128a^3+4a} \left(\frac{\rho(y,y')}{D^{s(J)}}\right)^{1/a} \sum_{S \geq s \geq s(J)} D^{(s(J)-s)/a} \inf_J M_{\mathcal{B},1}|g|.$$

Since $D^{-1/a} \leq \frac{1}{2}$, we have

$$\sum_{S \geq s \geq s(J)} D^{(s(J)-s)/a} \leq 2.$$

Estimate `TreeHolder`, and therefore the lemma, follow.

Lemma 1.7.5.2.7

We have for all $J \in \mathcal{J}'$ and all bounded g with bounded support

$$\sup_{B'(J)} |T^*_{\mathfrak{T}(\mathfrak{u}_2)\cap\mathfrak{S}} g| \leq \inf_{B^\circ(J)} |T^*_{\mathfrak{T}(\mathfrak{u}_2)} g| + 2^{129a^3} \inf_J M_{\mathcal{B},1}|g|.$$

Proof for Lemma 1.7.5.2.7

Proof. By Lemma 1.7.5.2.6

$$\sup_{B'(J)} |T^*_{\mathfrak{T}(\mathfrak{u}_2)\cap\mathfrak{S}} g| \leq \inf_{B^\circ(J)} |T^*_{\mathfrak{T}(\mathfrak{u}_2)\cap\mathfrak{S}} g| + 2^{128a^3+4a+3} \inf_J M_{\mathcal{B},1}|g|$$

$$\leq \inf_{B^\circ(J)} |T^*_{\mathfrak{T}(\mathfrak{u}_2)} g| + \sup_{B^\circ(J)} |T^*_{\mathfrak{T}(\mathfrak{u}_2)\setminus\mathfrak{S}} g| + 2^{128a^3+4a+3} \inf_J M_{\mathcal{B},1}|g|,$$

and by Lemma 1.7.5.2.4

$$\leq \inf_{B^\circ(J)} |T^*_{\mathfrak{T}(\mathfrak{u}_2)} g| + (2^{104a^3} + 2^{128a^3+4a+3}) \inf_J M_{\mathcal{B},1}|g|.$$

This completes the proof.

Proof for Lemma 1.7.5.2.1

Proof of Lemma 1.7.5.2.1. Let P be the product on the right-hand side of `hHolder`, and h_J be as defined in `def-hj`. By `eq-pao-2` and Lemma 1.7.4.1, the function h_J is supported in $B'(J) \cap \mathcal{I}(\mathfrak{u}_1)$. By `eq-pao-2`, Lemma 1.7.5.2.6 and Lemma 1.7.5.2.7, we have for all $y \in B'(J)$:

$$|h_J(y)| \leq 2^{257a^3+4a+3} P.$$

We have by the triangle inequality

$$|h_J(y) - h_J(y')|$$

$$\leq |\chi_J(y) - \chi_J(y')||T^*_{\mathfrak{T}(\mathfrak{u}_1)} g_1(y)||T^*_{\mathfrak{T}(\mathfrak{u}_2)\cap\mathfrak{S}} g_2(y)|$$

$$+|\chi_J(y')||e(\mathcal{Q}(\mathfrak{u}_1)(y))T^*_{\mathfrak{T}(\mathfrak{u}_1)} g_1(y) - e(\mathcal{Q}(\mathfrak{u}_1)(y'))T^*_{\mathfrak{T}(\mathfrak{u}_1)} g_1(y')||T^*_{\mathfrak{T}(\mathfrak{u}_2)\cap\mathfrak{S}} g_2(y)|$$

$$+|\chi_J(y')||T^*_{\mathfrak{T}(\mathfrak{u}_1)}g_1(y')||e(\mathcal{Q}(\mathfrak{u}_2)(y))T^*_{\mathfrak{T}(\mathfrak{u}_2)\cap\mathfrak{S}}g_2(y)-e(\mathcal{Q}(\mathfrak{u}_2)(y'))T^*_{\mathfrak{T}(\mathfrak{u}_2)\cap\mathfrak{S}}g_2(y')|.$$

As h_J is supported in $\mathcal{I}(\mathfrak{u}_1)$, we can assume without loss of generality that $y' \in \mathcal{I}(\mathfrak{u}_1)$. If $y \notin \mathcal{I}(\mathfrak{u}_1)$, then `eq-h-Lip-1` vanishes. If $y \in \mathcal{I}(\mathfrak{u}_1)$ then we have by `eq-pao-3`, Lemma 1.7.5.2.6 and Lemma 1.7.5.2.7 that `eq-h-Lip-1` is bounded by

$$2^{484a^3+4a+3}\frac{\rho(y,y')}{D^{s(J)}}P,$$

where P denotes the product on the right hand side of `hHolder`.By `eq-pao-2`, Lemma 1.7.5.2.6 and Lemma 1.7.5.2.7, we have the bound

$$2^{257a^3+4a+1}\left(\frac{\rho(y,y')}{D^{s(J)}}\right)^{1/a}P.$$

By `eq-pao-2`, and twice Lemma 1.7.5.2.6, we have the bound

$$2^{256a^3+8a+5}\left(\frac{\rho(y,y')}{D^{s(J)}}\right)^{1/a}P.$$

Using that $\rho(y,y') \le 32D^{s(J)}$ and $a \ge 4$, the lemma follows.

The van der Corput estimate

Lemma 1.7.5.3.1

For all $J \in \mathcal{J}'$, we have that

$$d_{B(J)}(\mathcal{Q}(\mathfrak{u}_1),\mathcal{Q}(\mathfrak{u}_2)) \ge 2^{-201a^3}2^{Zn/2}.$$

Proof for Lemma 1.7.5.3.1

Proof. Since $\emptyset \ne \mathfrak{T}(\mathfrak{u}_1) \subset \mathfrak{S}$ by Lemma 1.7.4.7, there exists at least one tile $\mathfrak{p} \in \mathcal{S}$ with $\mathcal{I}(\mathfrak{p}) \subsetneq \mathcal{I}(\mathfrak{u}_1)$. Thus $\mathcal{I}(\mathfrak{u}_1) \notin \mathcal{J}'$, so $J \subsetneq \mathcal{I}(\mathfrak{u}_1)$. Thus there exists a cube $J' \in \mathcal{D}$ with $J \subset J'$ and $s(J') = s(J)+1$, by `coverdyadic` and `dyadicproperty`. By definition of $\mathcal{J}'$ and the triangle inequality, there exists $\mathfrak{p} \in \mathfrak{S}$ such that

$$\mathcal{I}(\mathfrak{p}) \subset B(c(J'),100D^{s(J')+1}) \subset B(c(J),128D^{s(J)+2}).$$

Thus, by definition of $\mathfrak{S}$:

$$2^{Zn/2} \le d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{u}_1),\mathcal{Q}(\mathfrak{u}_2)) \le d_{B(c(J),128D^{s(J)+2})}(\mathcal{Q}(\mathfrak{u}_1),\mathcal{Q}(\mathfrak{u}_2)).$$

By the doubling property `firstdb`, this is

$$\le 2^{200a^3+4a}d_{B(J)}(\mathcal{Q}(\mathfrak{u}_1),\mathcal{Q}(\mathfrak{u}_2)),$$

which gives the lemma using $a \ge 4$.

Now we are ready to prove Lemma 1.7.4.5.

Proof for Lemma 1.7.4.5

Proof of Lemma 1.7.4.5. The left-hand side of `eq-lhs-big-sep-tree` equals

$$\left|\int_X T^*_{\mathfrak{T}(\mathfrak{u}_1)} g_1 \overline{T^*_{\mathfrak{T}(\mathfrak{u}_2)\cap\mathfrak{S}} g_2}\right|.$$

By Lemma 1.7.4.1, the right hand side is supported in $\mathcal{I}(\mathfrak{u}_1)$. Using `eq-pao-1` of Lemma 1.7.5.1.2 and the definition `def-hj` of h_J, we thus have

$$\le \sum_{J\in\mathcal{J}'}\left|\int_X e(\mathcal{Q}(\mathfrak{u}_2)(y)-\mathcal{Q}(\mathfrak{u}_1)(y))h_J(y)\,\mathrm{d}\mu(y)\right|.$$

Using Theorem 1.2.5 with the ball $B(J)$, we bound this by

$$\le 2^{7a}\sum_{J\in\mathcal{J}'}\mu(B(J))\|h_J\|_{C^\tau(B'(J))}(1+d_{B(J)}(\mathcal{Q}(\mathfrak{u}_2),\mathcal{Q}(\mathfrak{u}_1)))^{-1/(2a^2+a^3)}.$$

Using Lemma 1.7.5.2.1, Lemma 1.7.5.3.1 and $a\ge 4$, we bound the above by

$$\le 2^{485a^3+7a+3a^3+3a}2^{-Zn/(4a^2+2a^3)}\sum_{J\in\mathcal{J}'}\mu(B(J))\times\prod_{j=1}^{2}(\inf_{B^\circ(J)}|T^*_{\mathfrak{T}(\mathfrak{u}_j)}g_j|+\inf_J M_{\mathcal{B},1}g_j).$$

By the doubling property `doublingx`

$$\mu(B(J))\le 2^{6a}\mu(B^\circ(J)),$$

thus

$$\mu(B(J))\prod_{j=1}^{2}(\inf_{B^\circ(J)}|T^*_{\mathfrak{T}(\mathfrak{u}_j)}g_j|+\inf_J M_{\mathcal{B},1}g_j)$$

$$\le 2^{6a}\int_{B^\circ(J)}\prod_{j=1}^{2}(|T^*_{\mathfrak{T}(\mathfrak{u}_j)}g_j|(x)+M_{\mathcal{B},1}g_j(x))\,\mathrm{d}\mu(x)$$

$$\le 2^{6a}\int_J\prod_{j=1}^{2}(|T^*_{\mathfrak{T}(\mathfrak{u}_j)}g_j|(x)+M_{\mathcal{B},1}g_j(x))\,\mathrm{d}\mu(x).$$

Summing over $J\in\mathcal{J}'$, we obtain the estimate `eq-big-sep-1`:

$$2^{499a^3}2^{-Zn/(4a^2+2a^3)}\int_X\prod_{j=1}^{2}(|T^*_{\mathfrak{T}(\mathfrak{u}_j)}g_j|(x)+M_{\mathcal{B},1}g_j(x))\,\mathrm{d}\mu(x).$$

Applying the Cauchy-Schwarz inequality, Lemma 1.7.4.5 follows.

Proof of The Remaining Tiles Lemma

Lemma 1.7.6.1

We have

$$\mathcal{I}(\mathfrak{u}_1) = \dot{\bigcup_{J \in \mathcal{J}'}} J.$$

Proof for Lemma 1.7.6.1

Proof. By Lemma 1.7.1.2, it remains only to show that each $J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}_1))$ with $J \cap \mathcal{I}(\mathfrak{u}_1) \neq \emptyset$ is in $\mathcal{J}'$. But if $J \notin \mathcal{J}'$, then by `dyadicproperty` $\mathcal{I}(\mathfrak{u}_1) \subsetneq J$. Pick $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_1)$. Then $\mathcal{I}(\mathfrak{p}) \subsetneq J$. This contradicts the definition of $\mathcal{J}(\mathfrak{T}(\mathfrak{u}_1))$.

Lemma 1.7.4.6 follows from the following key estimate.

Lemma 1.7.6.2

We have for all bounded f with bounded support

$$\|P_{\mathcal{J}'}|T^*_{\mathfrak{T}(\mathfrak{u}_2)\setminus\mathfrak{S}} g_2|\|_2 \le 2^{102a^3+21a+5} 2^{-\frac{25}{101a} Zn\kappa} \|\mathbf{1}_{\mathcal{I}(\mathfrak{u}_1)} M_{\mathcal{B},1}|g_2|\|_2.$$

We prove this lemma below. First, we deduce Lemma 1.7.4.6.

Proof for Lemma 1.7.4.6

Proof of Lemma 1.7.4.6. By Lemma 1.7.2.1 and Lemma 1.7.4.1, we have that the left-hand side of `eq-lhs-small-sep-tree` is bounded by

$$2^{130a^3} \|P_{\mathcal{L}(\mathfrak{T}(\mathfrak{u}_1))}|\mathbf{1}_{\mathcal{I}(\mathfrak{u}_1)} g_1|\|_2 \|P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}_1))}|\mathbf{1}_{\mathcal{I}(\mathfrak{u}_1)} T^*_{\mathfrak{T}(\mathfrak{u}_2)\setminus\mathfrak{S}} g_2|\|_2.$$

It follows from the definition of the projection operator P and Jensen's inequality that

$$\|P_{\mathcal{L}(\mathfrak{T}(\mathfrak{u}_1))}|g_1 \mathbf{1}_{\mathcal{I}(\mathfrak{u}_1)}|\|_2 \le \|g_1 \mathbf{1}_{\mathcal{I}(\mathfrak{u}_1)}\|_2.$$

By Lemma 1.7.6.1, a cube $J \in \mathcal{J}(\mathfrak{T}(\mathfrak{u}_1))$ intersects $\mathcal{I}(\mathfrak{u}_1)$ if and only if $J \in \mathcal{J}'$. Thus

$$P_{\mathcal{J}(\mathfrak{T}(\mathfrak{u}_1))}|\mathbf{1}_{\mathcal{I}(\mathfrak{u}_1)} T^*_{\mathfrak{T}(\mathfrak{u}_2)\setminus\mathfrak{S}} g_2| = P_{\mathcal{J}'}|T^*_{\mathfrak{T}(\mathfrak{u}_2)\setminus\mathfrak{S}} g_2|.$$

Combining this with Lemma 1.7.6.2, the definition `definekappa` and $a \ge 4$ proves the lemma.

We need two more auxiliary lemmas before we prove Lemma 1.7.6.2.

Lemma 1.7.6.3

If $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_2) \setminus \mathfrak{S}$ and $J \in \mathcal{J}'$ with $B(\mathcal{I}(\mathfrak{p})) \cap B(J) \neq \emptyset$, then

$$\mathrm{s}(\mathfrak{p}) \le s(J) + 2 - \frac{Zn}{202a^3}.$$

Proof for Lemma 1.7.6.3

Proof. Suppose that $\mathrm{s}(\mathfrak{p}) > s(J) + 2 - \frac{Zn}{202a^3} =: s(J) - s_1$. Then, we have $s_1 + 2 \geq 0$ so

$$\rho(\mathrm{c}(\mathfrak{p}), c(J)) \leq 8D^{s(J)} + 8D^{\mathrm{s}(\mathfrak{p})} \leq 16D^{\mathrm{s}(\mathfrak{p})+s_1+2}.$$

There exists a tile $\mathfrak{q} \in \mathfrak{T}(\mathfrak{u}_1)$. By `forest1`, it satisfies $\mathcal{I}(\mathfrak{q}) \subsetneq \mathcal{I}(\mathfrak{u}_1)$. Thus $\mathcal{I}(\mathfrak{u}_1) \notin \mathcal{J}'$. It follows that $J \subsetneq \mathcal{I}(\mathfrak{u}_1)$. By `coverdyadic` and `dyadicproperty`, there exists a cube $J' \in \mathcal{D}$ with $J \subset J'$ and $s(J') = s(J) + 1$. By definition of $\mathcal{J}'$, there exists a tile $\mathfrak{p}' \in \mathfrak{T}(\mathfrak{u}_1)$ with

$$\mathcal{I}(\mathfrak{p}') \subset B(c(J'), 100D^{s(J')+1}).$$

By the triangle inequality, the definition `defineD` and $a \geq 4$, we have

$$B(c(J'), 100D^{s(J')+1}) \subset B(\mathrm{c}(\mathfrak{p}), 128D^{\mathrm{s}(\mathfrak{p})+s_1+2}).$$

Since $\mathfrak{p}' \in \mathfrak{T}(\mathfrak{u}_1)$ and $\mathcal{I}(\mathfrak{u}_1) \subset \mathcal{I}(\mathfrak{u}_2)$, we have by `forest5`

$$d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \mathcal{Q}(\mathfrak{u}_2)) > 2^{Z(n+1)}.$$

Hence, by `forest1`, the triangle inequality and using that by `defineZ` $Z(n+1) = 2^{12a}(n+1) \geq 3$

$$d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)) > 2^{Z(n+1)} - 4 \geq 2^{Z(n+1)-1}.$$

It follows that

$$2^{Z(n+1)-1} \leq d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)) \leq d_{B(\mathrm{c}(\mathfrak{p}), 128D^{\mathrm{s}(\mathfrak{p})+s_1+2})}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)).$$

Using `firstdb`, we obtain

$$\leq 2^{9a+100a^3(s_1+3)} d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{u}_1), \mathcal{Q}(\mathfrak{u}_2)).$$

Since $\mathfrak{p}' \notin \mathfrak{S}$ this is bounded by

$$\leq 2^{9a+100a^3(s_1+3)} 2^{Zn/2}.$$

Thus

$$Zn/2 + Z - 1 \leq 9a + 100a^3(s_1 + 3),$$

contradicting the definition of s_1.

Lemma 1.7.6.4

For each $J \in \mathcal{J}'$ and all s, we have

$$\frac{1}{\mu(J)} \int_J \Bigg(\sum_{\substack{I \in \mathcal{D}, s(I) = s(J) - s \\ I \cap \mathcal{I}(\mathfrak{u}_1) = \emptyset \\ J \cap B(I) \neq \emptyset}} \mathbf{1}_{B(I)} \Bigg)^2 \mathrm{d}\mu \leq 2^{14a+1} (8D^{-s})^{\kappa}.$$

Proof for Lemma 1.7.6.4

Proof. Since $J \in \mathcal{J}'$ we have $J \subset \mathcal{I}(\mathfrak{u}_1)$. Thus, if $B(I) \cap J \neq \emptyset$ then

$$B(I) \cap J \subset \{x \in J \ : \ \rho(x, X \setminus J) \leq 8D^{s(I)}\}.$$

Furthermore, for each s the balls $B(I)$ with $s(I) = s$ have bounded overlap: Consider the collection $\mathcal{D}_{s,x}$ of all $I \in \mathcal{D}$ with $x \in B(I)$ and $s(I) = s$. By `eq-vol-sp-cube` and `dyadicproperty`, the balls $B(c(I), \frac{1}{4}D^{s(I)})$, $I \in \mathcal{D}_{s,x}$ are disjoint, and by the triangle inequality, they are contained in $B(x, 9D^s)$. By the doubling property `doublingx`, we have

$$\mu(B(x, 9D^s)) \leq 2^{7a}\mu(B(c(I), \frac{1}{4}D^{s(I)}))$$

for each $I \in \mathcal{D}_{s,x}$. Thus

$$\mu(B(x, 9D^s)) \geq \sum_{I \in \mathcal{D}_{s,x}} \mu(B(c(I), \frac{1}{4}D^{s(I)})) \geq 2^{-7a}|\mathcal{D}_{s,x}|\mu(B(x, 9D^s)).$$

Dividing by the positive $\mu(B(x, 9D^s))$, we obtain that for each x

$$\left(\sum_{\substack{I \in \mathcal{D}, s(I) = s(J) - s \\ I \cap \mathcal{I}(\mathfrak{u}_1) = \emptyset \\ J \cap B(I) \neq \emptyset}} \mathbf{1}_{B(I)}(x) \right)^2 \leq |\mathcal{D}_{s(J)-s,x}|^2 \leq 2^{14a}.$$

Combining this with the previous inclusion and the small boundary property `eq-small-boundary`, noting that $8D^{s(I)} = 8D^{-s}D^{s(J)}$, the lemma follows.

Proof for Lemma 1.7.6.2

Proof of Lemma 1.7.6.2. Expanding the definition of $P_{\mathcal{J}'}$, we have

$$\|P_{\mathcal{J}'}|T^*_{\mathfrak{T}(\mathfrak{u}_2)\setminus\mathfrak{S}} g_2|\|_2$$

$$= \left(\sum_{J \in \mathcal{J}'} \frac{1}{\mu(J)} \left(\int_J \left| \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_2)\setminus\mathfrak{S}} T^*_{\mathfrak{p}} g_2(y) \right| \mathrm{d}\mu(y) \right)^2 \right)^{1/2}.$$

By Lemma 1.7.4.1, the innermost sum in the last display is 0 if $J \cap B(\mathcal{I}(\mathfrak{p})) = \emptyset$. Then we split that sum according to the scale of $\mathfrak{p}$ relative to the scale of J. By Lemma 1.7.6.3, $s_1 \leq s(J) - \mathrm{s}(\mathfrak{p}) \leq 2S$ with $s_1 := \lfloor \frac{Zn}{202a^3} - 2 \rfloor$

$$= \left(\sum_{J \in \mathcal{J}'} \frac{1}{\mu(J)} \left(\int_J \left| \sum_{s=s_1}^{2S} \sum_{\substack{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_2)\setminus\mathfrak{S} \\ \mathrm{s}(\mathfrak{p}) = s(J) - s \\ J \cap B(\mathcal{I}(\mathfrak{p})) \neq \emptyset}} T^*_{\mathfrak{p}} g_2(y) \right| \mathrm{d}\mu(y) \right)^2 \right)^{1/2}.$$

Then we apply the triangle inequality and Minkowski's inequality to get

$$\leq \sum_{s=s_1}^{2S} \left(\sum_{J \in \mathcal{J}'} \frac{1}{\mu(J)} \left(\int_J \sum_{\substack{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_2) \setminus \mathfrak{S} \\ \mathrm{s}(\mathfrak{p}) = s(J) - s \\ J \cap B(\mathcal{I}(\mathfrak{p})) \neq \emptyset}} |T_{\mathfrak{p}}^* g_2(y)| \, \mathrm{d}\mu(y) \right)^2 \right)^{1/2} .$$

We have by Lemma 1.7.4.1 and `eq-Ks-size`

$$|T_{\mathfrak{p}}^* g_2(y)| \leq 2^{102a^3} \mathbf{1}_{B(\mathrm{c}(\mathfrak{p}), 8D^{\mathrm{s}(\mathfrak{p})})}(y) \int_{E(\mathfrak{p})} \frac{1}{\mu(B(x, D^{\mathrm{s}(\mathfrak{p})}))} |g_2(x)| \, \mathrm{d}\mu(x).$$

Using the doubling property `doublingx`, it follows that

$$\mu(B(\mathrm{c}(\mathfrak{p}), 8D^{\mathrm{s}(\mathfrak{p})})) \leq 2^{4a} \mu(B(x, D^{\mathrm{s}(\mathfrak{p})})).$$

Thus, using also $a \geq 4$

$$|T_{\mathfrak{p}}^* g_2(y)| \leq 2^{102a^3 + 4a} \mathbf{1}_{B(\mathrm{c}(\mathfrak{p}), 8D^{\mathrm{s}(\mathfrak{p})})}(y) \frac{1}{\mu(B(\mathrm{c}(\mathfrak{p}), 8D^{\mathrm{s}(\mathfrak{p})}))} \int_{E(\mathfrak{p})} |g_2(x)| \, \mathrm{d}\mu(x).$$

Since for each $I \in \mathcal{D}$ the sets $E(\mathfrak{p})$, $\mathfrak{p} \in \mathfrak{P}(I)$ are disjoint, it follows that

$$\int_J \sum_{\substack{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_2) \setminus \mathfrak{S} \\ \mathcal{I}(\mathfrak{p}) = I \\ J \cap B(\mathcal{I}(\mathfrak{p})) \neq \emptyset}} |T_{\mathfrak{p}}^* g_2(y)| \, \mathrm{d}\mu$$

$$\leq 2^{102a^3 + 4a} \int_J \mathbf{1}_{B(I)} \frac{1}{\mu(B(\mathrm{c}(\mathfrak{p}), 8D^{\mathrm{s}(\mathfrak{p})}))} \int_{B(\mathrm{c}(\mathfrak{p}), 8D^{\mathrm{s}(\mathfrak{p})})} |g_2(x)| \, \mathrm{d}\mu(x)$$

$$\leq 2^{102a^3 + 4a} \int_J M_{\mathcal{B},1} |g_2|(y) \mathbf{1}_{B(I)}(y) \, \mathrm{d}\mu(y).$$

By Lemma 1.7.4.7, we have $\mathcal{I}(\mathfrak{p}) \cap \mathcal{I}(\mathfrak{u}_1) = \emptyset$ for all $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u}_2) \setminus \mathfrak{S}$. Thus we can estimate `eq-sep-tree-small-1` by

$$2^{102a^3 + 4a} \sum_{s=s_1}^{2S} \left(\sum_{J \in \mathcal{J}'} \frac{1}{\mu(J)} \left(\int_J \sum_{\substack{I \in \mathcal{D}, s(I) = s(J) - s \\ I \cap \mathcal{I}(\mathfrak{u}_1) = \emptyset \\ J \cap B(I) \neq \emptyset}} M_{\mathcal{B},1} |g_2| \mathbf{1}_{B(I)} \, \mathrm{d}\mu \right)^2 \right)^{1/2} ,$$

which is by Cauchy-Schwarz at most

$$2^{102a^3 + 4a} \sum_{s=s_1}^{2S} \left(\sum_{J \in \mathcal{J}'} \int_J (M_{\mathcal{B},1} |g_2|)^2 \, \mathrm{d}\mu \times \frac{1}{\mu(J)} \int_J \left(\sum_{\substack{I \in \mathcal{D}, s(I) = s(J) - s \\ I \cap \mathcal{I}(\mathfrak{u}_1) = \emptyset \\ J \cap B(I) \neq \emptyset}} \mathbf{1}_{B(I)} \right)^2 \mathrm{d}\mu \right)^{1/2} .$$

Using Lemma 1.7.6.4, we bound this by

$$2^{102a^3+4a}\sum_{s=s_1}^{2S}\left(2^{14a+1}(8D^{-s})^{\kappa}\sum_{J\in\mathcal{J}'}\int_J (M_{\mathcal{B},1}|g_2|)^2\right)^{1/2},$$

and since dyadic cubes in $\mathcal{J}'$ form a partition of $\mathcal{I}(\mathfrak{u}_1)$ by Lemma 1.7.6.1, $\kappa \le 1$ by `definekappa`, and $a \ge 4$

$$\le 2^{102a^3+11a+2}\sum_{s=s_1}^{2S} D^{-s\kappa/2}\|\mathbf{1}_{\mathcal{I}(\mathfrak{u}_1)}M_{\mathcal{B},1}|g_2|\|_2$$

$$\le 2^{102a^3+11a+2}D^{-s_1\kappa/2}\frac{1}{1-D^{-\kappa/2}}\|\mathbf{1}_{\mathcal{I}(\mathfrak{u}_1)}M_{\mathcal{B},1}|g_2|\|_2.$$

By convexity of $t\mapsto D^{-t}$ and since $D\ge 2$, we have for all $0\le t\le 1$

$$D^{-t}\le 1-t(1-1/D)\le 1-\frac{1}{2}t.$$

Using this for $t=\kappa/2$ and using that $s_1=\frac{Zn}{202a^3}-2$ and the definitions `defineD` and `definekappa` of κ and D

$$\le 2^{102a^3+11a+2}2^{-100a^2(\frac{Zn}{202a^3}-3)\frac{\kappa}{2}}\frac{2}{\kappa}\|\mathbf{1}_{\mathcal{I}(\mathfrak{u}_1)}M_{\mathcal{B},1}|g_2|\|_2$$

$$= 2^{102a^3+21a+4}2^{150a^2\kappa}2^{-\frac{100}{404a}Zn\kappa}\|\mathbf{1}_{\mathcal{I}(\mathfrak{u}_1)}M_{\mathcal{B},1}|g_2|\|_2.$$

Using the definition `definekappa` of κ and $a\ge 4$, the lemma follows.

Forests

In this subsection, we complete the proof of Theorem 1.2.4 from the results of the previous subsections.

Define an n-row to be an n-forest $(\mathfrak{U},\mathfrak{T})$, i.e. satisfying conditions `forest1` - `forest6`, such that in addition the sets $\mathcal{I}(\mathfrak{u})$, $\mathfrak{u}\in\mathfrak{U}$ are pairwise disjoint.

Lemma 1.7.7.1

Let $(\mathfrak{U},\mathfrak{T})$ be an n-forest. Then there exists a decomposition

$$\mathfrak{U}=\dot{\bigcup_{1\le j\le 2^n}}\mathfrak{U}_j$$

such that for all $j=1,\dots,2^n$ the pair $(\mathfrak{U}_j,\mathfrak{T}|_{\mathfrak{U}_j})$ is an n-row.

Proof for Lemma 1.7.7.1

Proof. Define recursively $\mathfrak{U}_j$ to be a maximal disjoint set of tiles $\mathfrak{u}$ in

$$\mathfrak{U} \setminus \bigcup_{j'<j} \mathfrak{U}_{j'}$$

with inclusion maximal $\mathcal{I}(\mathfrak{u})$. Properties `forest1`, -`forest6` for $(\mathfrak{U}_j, \mathfrak{T}|_{\mathfrak{U}_k})$ follow immediately from the corresponding properties for $(\mathfrak{U}, \mathfrak{T})$, and the cubes $\mathcal{I}(\mathfrak{u})$, $\mathfrak{u} \in \mathfrak{U}_j$ are disjoint by definition. The collections $\mathfrak{U}_j$ are also disjoint by definition.Now we show by induction on j that each point is contained in at most $2^n - j$ cubes $\mathcal{I}(\mathfrak{u})$ with $\mathfrak{u} \in \mathfrak{U} \setminus \bigcup_{j' \le j} \mathfrak{U}_{j'}$. This implies that $\bigcup_{j=1}^{2^n} \mathfrak{U}_j = \mathfrak{U}$, which completes the proof of the Lemma. For $j = 0$ each point is contained in at most 2^n cubes by `forest3`. For larger j, if x is contained in any cube $\mathcal{I}(\mathfrak{u})$ with $\mathfrak{u} \in \mathfrak{U} \setminus \bigcup_{j'<j} \mathfrak{U}_{j'}$, then it is contained in a maximal such cube. Thus it is contained in a cube in $\mathcal{I}(\mathfrak{u})$ with $\mathfrak{u} \in \mathfrak{U}_j$. Thus the number $\mathfrak{u} \in \mathfrak{U} \setminus \bigcup_{j' \le j} \mathfrak{U}_{j'}$ with $x \in \mathcal{I}(\mathfrak{u})$ is zero, or is less than the number of $\mathfrak{u} \in \mathfrak{U} \setminus \bigcup_{j' \le j-1} \mathfrak{U}_{j'}$ with $x \in \mathcal{I}(\mathfrak{u})$ by at least one.

We pick a decomposition of the forest $(\mathfrak{U}, \mathfrak{T})$ into 2^n n-rows

$$(\mathfrak{U}_j, \mathfrak{T}_j) := (\mathfrak{U}_j, \mathfrak{T}|_{\mathfrak{U}_j})$$

as in Lemma 1.7.7.1. To save some space in the proofs of the remaining lemmas in this section we will write

$$T_{\mathfrak{C}} = \sum_{\mathfrak{p} \in \mathfrak{C}} T_{\mathfrak{p}}, \qquad T^*_{\mathfrak{C}} = \sum_{\mathfrak{p} \in \mathfrak{C}} T^*_{\mathfrak{p}},$$

$$T_{\mathfrak{R}_j} = \sum_{\mathfrak{u} \in \mathfrak{U}_j} T_{\mathfrak{T}(\mathfrak{u})}, \qquad T^*_{\mathfrak{R}_j} = \sum_{\mathfrak{u} \in \mathfrak{U}_j} T^*_{\mathfrak{T}(\mathfrak{u})}.$$

Lemma 1.7.7.2

For each $1 \le j \le 2^n$ and each bounded g supported on G we have

$$\left\| T^*_{\mathfrak{R}_j} g \right\|_2 \le 2^{182a^3} 2^{-n/2} \|g\|_2$$

and

$$\left\| \mathbf{1}_F T^*_{\mathfrak{R}_j} g \right\|_2 \le 2^{283a^3} 2^{-n/2} \operatorname{dens}_2\Big(\bigcup_{\mathfrak{u} \in \mathfrak{U}} \mathfrak{T}(\mathfrak{u})\Big)^{1/2} \|g\|_2.$$

Proof for Lemma 1.7.7.2

Proof. Since for each j the top cubes $\mathcal{I}(\mathfrak{u})$, $\mathfrak{u} \in \mathfrak{U}_j$ are disjoint, we have for all bounded g supported on G by Lemma 1.7.4.1

$$\left\| \mathbf{1}_F \sum_{\mathfrak{u} \in \mathfrak{U}_j} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T^*_{\mathfrak{p}} g \right\|_2^2 = \left\| \mathbf{1}_F \sum_{\mathfrak{u} \in \mathfrak{U}_j} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} \mathbf{1}_{\mathcal{I}(\mathfrak{u})} T^*_{\mathfrak{p}} \mathbf{1}_{\mathcal{I}(\mathfrak{u})} g \right\|_2^2$$

$$= \sum_{\mathfrak{u}\in\mathfrak{U}_j} \int_{\mathcal{I}(\mathfrak{u})} \left| \mathbf{1}_F \sum_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}}^* \mathbf{1}_{\mathcal{I}(\mathfrak{u})} g \right|^2 \, \mathrm{d}\mu \le \sum_{\mathfrak{u}\in\mathfrak{U}_j} \left\| \mathbf{1}_F \sum_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}}^* \mathbf{1}_G \mathbf{1}_{\mathcal{I}(\mathfrak{u})} g \right\|_2^2 .$$

Applying Lemma 1.7.4.2 and the density assumption `forest4`, then taking square roots, we obtain

$$\left\| \mathbf{1}_F T_{\mathfrak{R}_j}^* g \right\|_2 \le 2^{282a^3} 2^{(4a+1-n)/2} \operatorname{dens}_2\Big(\bigcup_{\mathfrak{u}\in\mathfrak{U}} \mathfrak{T}(\mathfrak{u})\Big)^{1/2} \left(\sum_{\mathfrak{u}\in\mathfrak{U}_j} \left\| \mathbf{1}_{\mathcal{I}(\mathfrak{u})} g \right\|_2^2 \right)^{1/2} .$$

Again by disjointedness of the cubes $\mathcal{I}(\mathfrak{u})$, this is estimated by

$$2^{282a^3} 2^{(4a+1-n)/2} \operatorname{dens}_2\Big(\bigcup_{\mathfrak{u}\in\mathfrak{U}} \mathfrak{T}(\mathfrak{u})\Big)^{1/2} \|g\|_2 .$$

Thus `eq-row-bound-2` follows, since $a \ge 4$. The proof of `eq-row-bound-1` is the same up to replacing F by X.

Lemma 1.7.7.3

For all $1 \le j, j' \le 2^n$ with $j \ne j'$ and for all bounded g_1, g_2 supported on G, it holds that

$$\left| \int T_{\mathfrak{R}_j}^* g_1 \overline{T_{\mathfrak{R}_{j'}}^* g_2} \, \mathrm{d}\mu \right| \le 2^{876a^3 - 4n} \|g_1\|_2 \|g_2\|_2 .$$

Proof for Lemma 1.7.7.3

Proof. We have by Lemma 1.7.4.1 and the triangle inequality that

$$\left| \int T_{\mathfrak{R}_j}^* g_1 \overline{T_{\mathfrak{R}_{j'}}^* g_2} \, \mathrm{d}\mu \right|$$

$$\le \sum_{\mathfrak{u}\in\mathfrak{U}_j} \sum_{\mathfrak{u}'\in\mathfrak{U}_{j'}} \left| \int T_{\mathfrak{T}_j(\mathfrak{u})}^* (\mathbf{1}_{\mathcal{I}(\mathfrak{u})} g_1) \overline{T_{\mathfrak{T}_{j'}(\mathfrak{u}')}^* (\mathbf{1}_{\mathcal{I}(\mathfrak{u}')} g_2)} \, \mathrm{d}\mu \right| .$$

By Lemma 1.7.4.4, this is bounded by

$$2^{512a^3 - 4n} \sum_{\mathfrak{u}\in\mathfrak{U}_j} \sum_{\mathfrak{u}'\in\mathfrak{U}_{j'}} \|S_{2,\mathfrak{u}}(\mathbf{1}_{\mathcal{I}(\mathfrak{u})} g_1)\|_{L^2(\mathcal{I}(\mathfrak{u}')\cap\mathcal{I}(\mathfrak{u}))} \|S_{2,\mathfrak{u}'}(\mathbf{1}_{\mathcal{I}(\mathfrak{u}')} g_2)\|_{L^2(\mathcal{I}(\mathfrak{u}')\cap\mathcal{I}(\mathfrak{u}))} .$$

We apply the Cauchy-Schwarz inequality in the form

$$\sum_{i\in M} a_i b_i \le \Big(\sum_{i\in M} a_i^2\Big)^{1/2} \Big(\sum_{i\in M} b_i^2\Big)^{1/2}$$

to the outer two sums:

$$\leq 2^{512a^3-4n}\left(\sum_{\mathfrak{u}\in\mathfrak{U}_j}\sum_{\mathfrak{u}'\in\mathfrak{U}_{j'}}\|S_{2,\mathfrak{u}}(\mathbf{1}_{\mathcal{I}(\mathfrak{u})}g_1)\|^2_{L^2(\mathcal{I}(\mathfrak{u}')\cap\mathcal{I}(\mathfrak{u}))}\right)^{1/2}$$

$$\left(\sum_{\mathfrak{u}\in\mathfrak{U}_j}\sum_{\mathfrak{u}'\in\mathfrak{U}_{j'}}\|S_{2,\mathfrak{u}'}(\mathbf{1}_{\mathcal{I}(\mathfrak{u}')}g_2)\|^2_{L^2(\mathcal{I}(\mathfrak{u}')\cap\mathcal{I}(\mathfrak{u}))}\right)^{1/2}.$$

We can now estimate the factor involving g_1 as follows:

$$\sum_{\mathfrak{u}\in\mathfrak{U}_j}\sum_{\mathfrak{u}'\in\mathfrak{U}_{j'}}\|S_{2,\mathfrak{u}}(\mathbf{1}_{\mathcal{I}(\mathfrak{u})}g_1)\|^2_{L^2(\mathcal{I}(\mathfrak{u}')\cap\mathcal{I}(\mathfrak{u}))}$$

$$=\sum_{\mathfrak{u}\in\mathfrak{U}_j}\sum_{\mathfrak{u}'\in\mathfrak{U}_{j'}}\int_{\mathcal{I}(\mathfrak{u})\cap\mathcal{I}(\mathfrak{u}')}|S_{2,\mathfrak{u}}(\mathbf{1}_{\mathcal{I}(\mathfrak{u})}(y)g_1(y))|^2\,\mathrm{d}\mu(y).$$

By pairwise disjointedness of the sets $\mathcal{I}(\mathfrak{u}')$ for $\mathfrak{u}'\in\mathfrak{U}_{j'}$, we have

$$\leq\sum_{\mathfrak{u}\in\mathfrak{U}_j}\int_{\mathcal{I}(\mathfrak{u})}|S_{2,\mathfrak{u}}(\mathbf{1}_{\mathcal{I}(\mathfrak{u})}(y)g_1(y))|^2\,\mathrm{d}\mu(y)\leq\sum_{\mathfrak{u}\in\mathfrak{U}_j}\int_X|S_{2,\mathfrak{u}}(\mathbf{1}_{\mathcal{I}(\mathfrak{u})}(y)g_1(y))|^2\,\mathrm{d}\mu(y)$$

$$=\sum_{\mathfrak{u}\in\mathfrak{U}_j}\|S_{2,\mathfrak{u}}(\mathbf{1}_{\mathcal{I}(\mathfrak{u})}g_1)\|_2^2.$$

By Lemma 1.7.4.3 we now estimate:

$$\leq\sum_{\mathfrak{u}\in\mathfrak{U}_j}(2^{182a^3})^2\|\mathbf{1}_{\mathcal{I}(\mathfrak{u})}g_1\|_2^2.$$

By pairwise disjointedness of the sets $\mathcal{I}(\mathfrak{u})$ for $\mathfrak{u}\in\mathfrak{U}_j$ (and writing out the definition of L^2-norms), we have

$$\leq(2^{182a^3})^2\|g_1\|_2^2.$$

Arguing similarly for g_2, we obtain the desired inequality.

Define for $1\leq j\leq 2^n$

$$E_j:=\bigcup_{\mathfrak{u}\in\mathfrak{U}_j}\bigcup_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})}E(\mathfrak{p}).$$

Lemma 1.7.7.4

The sets E_j, $1\leq j\leq 2^n$ are pairwise disjoint.

Proof for Lemma 1.7.7.4

Proof. Suppose that $\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})$ and $\mathfrak{p}' \in \mathfrak{T}(\mathfrak{u}')$ with $\mathfrak{u} \neq \mathfrak{u}'$ and $x \in E(\mathfrak{p}) \cap E(\mathfrak{p}')$. Suppose without loss of generality that $\mathrm{s}(\mathfrak{p}) \leq \mathrm{s}(\mathfrak{p}')$. Then $x \in \mathcal{I}(\mathfrak{p}) \cap \mathcal{I}(\mathfrak{p}') \subset \mathcal{I}(\mathfrak{u}')$. By `dyadicproperty` it follows that $\mathcal{I}(\mathfrak{p}) \subset \mathcal{I}(\mathfrak{u}')$. By `forest5`, it follows that

$$d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{u}')) > 2^{Z(n+1)}.$$

By the triangle inequality. Lemma 1.2.1.2 and `forest1` it follows that

$$\begin{aligned} d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{p}')) &\geq d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), \mathcal{Q}(\mathfrak{u}')) - d_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}'), \mathcal{Q}(\mathfrak{u}')) \\ &> 2^{Z(n+1)} - d_{\mathfrak{p}'}(\mathcal{Q}(\mathfrak{p}'), \mathcal{Q}(\mathfrak{u}')) \\ &\geq 2^{Z(n+1)} - 4. \end{aligned}$$

Since $Z \geq 3$ by `defineZ`, it follows that $\mathcal{Q}(\mathfrak{p}') \notin B_{\mathfrak{p}}(\mathcal{Q}(\mathfrak{p}), 1)$, so $\Omega(\mathfrak{p}') \not\subset \Omega(\mathfrak{p})$ by `eq-freq-comp-ball`. Hence, by `eq-freq-dyadic`, $\Omega(\mathfrak{p}) \cap \Omega(\mathfrak{p}') = \emptyset$. But if $x \in E(\mathfrak{p}) \cap E(\mathfrak{p}')$ then $Q(x) \in \Omega(\mathfrak{p}) \cap \Omega(\mathfrak{p}')$. This is a contradiction, and the lemma follows.

Now we prove Theorem 1.2.4.

Proof for Theorem 1.2.4

Proof of Theorem 1.2.4. By `definetp*`, we have for each j

$$T^*_{\mathfrak{R}_j} g = \sum_{\mathfrak{u} \in \mathfrak{U}_j} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T^*_{\mathfrak{p}} g = \sum_{\mathfrak{u} \in \mathfrak{U}_j} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T^*_{\mathfrak{p}} \mathbf{1}_{E_j} g = T^*_{\mathfrak{R}_j} \mathbf{1}_{E_j} g.$$

Hence, by Lemma 1.7.7.1 and the triangle inequality,

$$\begin{aligned} \left\| \sum_{\mathfrak{u} \in \mathfrak{U}} \sum_{\mathfrak{p} \in \mathfrak{T}(\mathfrak{u})} T^*_{\mathfrak{p}} g \right\|_2^2 &= \left\| \sum_{j=1}^{2^n} T^*_{\mathfrak{R}_j} g \right\|_2^2 = \left\| \sum_{j=1}^{2^n} T^*_{\mathfrak{R}_j} \mathbf{1}_{E_j} g \right\|_2^2 \\ &= \int_X \left| \sum_{j=1}^{2^n} T^*_{\mathfrak{R}_j} \mathbf{1}_{E_j} g \right|^2 \, \mathrm{d}\mu \\ &\leq \sum_{j=1}^{2^n} \|T^*_{\mathfrak{R}_j} \mathbf{1}_{E_j} g\|_2^2 + \sum_{j=1}^{2^n} \sum_{\substack{j'=1 \\ j' \neq j}}^{2^n} \left| \int_X \overline{T^*_{\mathfrak{R}_j} \mathbf{1}_{E_j} g} T^*_{\mathfrak{R}_{j'}} \mathbf{1}_{E_{j'}} g \right| \, \mathrm{d}\mu \, . \end{aligned}$$

We use Lemma 1.7.7.2 to estimate each term in the first sum, and Lemma 1.7.7.3 to bound each term in the second sum:

$$\leq 2^{566a^3 - n} \sum_{j=1}^{2^n} \|\mathbf{1}_{E_j} g\|_2^2 + 2^{876a^3 - 4n} \sum_{j=1}^{2^n} \sum_{j'=1}^{2^n} \|\mathbf{1}_{E_j} g\|_2 \|\mathbf{1}_{E_{j'}} g\|_2.$$

By Cauchy-Schwarz in the second two sums, this is at most

$$2^{876a^3}(2^{-n}+2^n2^{-4n})\sum_{j=1}^n \|\mathbf{1}_{E_j}g\|_2^2,$$

and by disjointedness of the sets E_j, this is at most

$$2^{877a^3-n}\|g\|_2^2.$$

Taking square roots, it follows that for all g

$$\left\|\sum_{\mathfrak{u}\in\mathfrak{U}}\sum_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}}^* g\right\|_2 \le 2^{439a^3-\frac{n}{2}}\|g\|_2.$$

On the other hand, we have by disjointedness of the sets E_j from Lemma 1.7.7.4 and the triangle inequality

$$\left\|\mathbf{1}_G\sum_{\mathfrak{u}\in\mathfrak{U}}\sum_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f\right\|_2^2 = \left\|\sum_{j=1}^{2^n}\mathbf{1}_{E_j}\mathbf{1}_G T_{\mathfrak{R}_j} f\right\|_2^2 \le \sum_{j=1}^{2^n}\|\mathbf{1}_{E_j}\mathbf{1}_G T_{\mathfrak{R}_j} f\|_2^2 \le \sum_{j=1}^{2^n}\|\mathbf{1}_G T_{\mathfrak{R}_j} f\|_2^2.$$

Now with $|f| \le \mathbf{1}_F$ and Lemma 1.7.7.2 we obtain $\|\mathbf{1}_G T_{\mathfrak{R}_j} f\|_2^2 = \left|\int_X \overline{\mathbf{1}_G T_{\mathfrak{R}_j} f} T_{\mathfrak{R}_j} f\right| = \left|\int_X \overline{T_{\mathfrak{R}_j}^* \mathbf{1}_G T_{\mathfrak{R}_j} f}\mathbf{1}_F f\right| \le \|f\|_2\|\mathbf{1}_F T_{\mathfrak{R}_j}^*\mathbf{1}_G T_{\mathfrak{R}_j} f\|_2 \le 2^{283a^3-n/2}\,\mathrm{dens}_2(\bigcup_{\mathfrak{u}\in\mathfrak{U}}\mathfrak{T}(\mathfrak{u}))^{1/2}\|\mathbf{1}_G T_{\mathfrak{R}_j} f\|_2\|f\|_2$. Dividing this last inequality by the finite $\|\mathbf{1}_G T_{\mathfrak{R}_j} f\|_2$, substituting back and taking square roots we get $\left\|\mathbf{1}_G\sum_{\mathfrak{u}\in\mathfrak{U}}\sum_{\mathfrak{p}\in\mathfrak{T}(\mathfrak{u})} T_{\mathfrak{p}} f\right\|_2 \le 2^{283a^3}\,\mathrm{dens}_2(\bigcup_{\mathfrak{u}\in\mathfrak{U}}\mathfrak{T}(\mathfrak{u}))^{\frac{1}{2}}2^{-\frac{n}{2}}(\sum_{j=1}^{2^n}\|f\|_2^2)^{\frac{1}{2}} = 2^{283a^3}\,\mathrm{dens}_2(\bigcup_{\mathfrak{u}\in\mathfrak{U}}\mathfrak{T}(\mathfrak{u}))^{\frac{1}{2}}\|f\|_2$. Theorem 1.2.4 follows by taking the product of the $(2-\frac{2}{q})$-th power of `eq-forest-bound-1` and the $(\frac{2}{q}-1)$-st power of `eq-forest-bound-2`.

1.8 Proof of the H''older cancellative condition

We need the following auxiliary lemma. Recall that $\tau = 1/a$.
Lemma 1.8.1

Let $z \in X$ and $R > 0$. Let $\varphi : X \to \mathbb{C}$ be a function supported in the ball $B := B(z,R)$ with finite norm $\|\varphi\|_{C^\tau(B(z,2R))}$. Let $0 < t \le 1$. There exists a function $\tilde\varphi : X \to \mathbb{C}$, supported in $B(z,2R)$, such that for every $x \in X$

$$|\varphi(x)-\tilde\varphi(x)| \le (t/2)^\tau\|\varphi\|_{C^\tau(B(z,2R))}$$

and

$$\|\tilde\varphi\|_{\mathrm{Lip}(B(z,2R))} \le 2^{4a}t^{-1-a}\|\varphi\|_{C^\tau(B(z,2R))}.$$

Proof for Lemma 1.8.1

Proof. Define for $x, y \in X$ the Lipschitz and thus measurable function

$$L(x,y) := \max\{0, 1 - \frac{\rho(x,y)}{tR}\}.$$

We have that $L(x,y) \neq 0$ implies

$$y \in B(x, tR).$$

We have for $y \in B(x, 2^{-1}tR)$ that

$$|L(x,y)| \geq 2^{-1}.$$

Hence

$$\int L(x,y)\,\mathrm{d}\mu(y) \geq 2^{-1}\mu(B(x, 2^{-1}tR)).$$

Let n be the smallest integer so that

$$2^n t \geq 1.$$

Iterating $n+2$ times the doubling condition `doublingx`, we obtain

$$\int L(x,y)\,\mathrm{d}\mu(y) \geq 2^{-1-a(n+2)}\mu(B(x, 2R)).$$

Now define

$$\tilde{\varphi}(x) := \left(\int L(x,y)\,\mathrm{d}\mu(y)\right)^{-1} \int L(x,y)\varphi(y)\,\mathrm{d}\mu(y).$$

Using that φ is supported in $B(z, R)$ and `eql01`, we have that $\tilde{\varphi}$ is supported in $B(z, 2R)$.We prove `eq-firstt`. For any $x \in X$, using that L is nonnegative,

$$\left(\int L(x,y)\,\mathrm{d}\mu(y)\right) |\varphi(x) - \tilde{\varphi}(x)|$$

$$= \left|\int L(x,y)(\varphi(x) - \varphi(y))\,\mathrm{d}\mu(y)\right|.$$

Using `eql01`, we estimate the last display by

$$\leq \int_{B(x,tR)} L(x,y)|\varphi(x) - \varphi(y)|\,\mathrm{d}\mu(y).$$

We claim that in this integral, $|\varphi(x) - \varphi(y)| \leq \rho(x,y)^\tau \|\varphi\|_{C^\tau(B(z,2R))}(2R)^{-\tau}$. Indeed, if x or y does not belong to $B(z, 2R)$, then the other point is not in $B(z, R)$ as $\rho(x,y) \leq tR \leq R$. Therefore, $\varphi(x) = \varphi(y) =$

0 since φ is supported in $B(z,R)$. Otherwise, both points are in $B(z,2R)$, and the inequality follows from the definition of $\|\varphi\|_{C^\tau(B(z,2R))}$.Therefore, we can estimate the last display further by

$$\le \left(\int_{B(x,tR)} L(x,y)\rho(x,y)^\tau \,\mathrm{d}\mu(y)\right) \|\varphi\|_{C^\tau(B(z,2R))}(2R)^{-\tau}.$$

Using the condition on the domain of integration to estimate $\rho(x,y)$ by tR and then expanding the domain by positivity of the integrand, we estimate this further by

$$\le \left(\int L(x,y)\,\mathrm{d}\mu(y)\right) \|\varphi\|_{C^\tau(B(z,2R))}(t/2)^\tau.$$

Dividing the string of inequalities from `eql1` to `eql5` by the positive integral of L proves `eq-firstt`.We turn to `eq-secondt`. For every $x \in X$, we have

$$\left|\int L(x,y)\,\mathrm{d}\mu(y)\right| |\tilde\varphi(x)| = \left|\int L(x,y)\varphi(y)\,\mathrm{d}\mu(y)\right|$$

$$\le \left|\int L(x,y)\,\mathrm{d}\mu(y)\right| \sup_{x'\in X} |\varphi(x')|.$$

As φ is supported on B, dividing by the integral of L, we obtain

$$|\tilde\varphi(x)| \le \sup_{x'\in B} |\varphi(x')| \le \|\varphi\|_{C^\tau(B(z,2R))}.$$

If $\rho(x,x') \ge R$, then we have by the triangle inequality

$$R\frac{|\tilde\varphi(x') - \tilde\varphi(x)|}{\rho(x,x')} \le 2 \sup_{x''\in X} |\tilde\varphi(x'')| \le 2\|\varphi\|_{C^\tau(B(z,2R))}.$$

Now assume $\rho(x,x') < R$. For $y \in X$ we have by the triangle inequality and a two fold case distinction for the maximum in the definition of L,

$$|L(x,y) - L(x',y)| \le \frac{\rho(x,x')}{tR}.$$

We compute with `eql10`, first adding and subtracting a term in the integral,

$$\left(\int L(x,y)\,\mathrm{d}\mu(y)\right) |\tilde\varphi(x') - \tilde\varphi(x)| =$$

$$\left|\int L(x,y)\tilde\varphi(x') - L(x,y)\tilde\varphi(x) + L(x',y)\tilde\varphi(x') - L(x',y)\tilde\varphi(x')\,\mathrm{d}\mu(y)\right|.$$

Grouping the second and third and the first and fourth term, we obtain using the definition of $\tilde{\varphi}$ and Fubini,

$$\leq \left| \int (L(x', y) - L(x, y))\varphi(y) \, \mathrm{d}\mu(y) \right|$$

$$+ \left| \int L(x, y) \, \mathrm{d}\mu(y) - \int L(x', y) \, \mathrm{d}\mu(y) \right| |\tilde{\varphi}(x')|$$

$$\leq 2 \int |L(x, y) - L(x', y)| \, \mathrm{d}\mu(y) \|\varphi\|_{C^\tau(B(z,2R))},$$

where in the last inequality we have used `eql42`. Using further `eql10` and the support of L, we estimate the last display by

$$\leq 2\frac{\rho(x, x')}{tR}\mu(B(x, tR) \cup B(x', tR))\|\varphi\|_{C^\tau(B(z,2R))}.$$

Using $\rho(x, x') < R$ and the triangle inequality, we estimate the last display by

$$\leq 2\frac{\rho(x, x')}{tR}\mu(B(x, 2R))\|\varphi\|_{C^\tau(B(z,2R))}.$$

Dividing by the integral over L and using `eql32` and `2nt1`, we obtain

$$\frac{R|\tilde{\varphi}(x') - \tilde{\varphi}(x)|}{\rho(x, x')} \leq 2^{2+a(n+2)}t^{-1}\|\varphi\|_{C^\tau(B(z,2R))} \leq 2^{2+3a}t^{-1-a}\|\varphi\|_{C^\tau(B(z,2R))}.$$

Combining `eql52` and `eql226` using $a \geq 4$ and $t \leq 1$ and adding `eql42` proves `eq-secondt` and completes the proof of Lemma 1.8.1.

We turn to the proof of Theorem 1.2.5.

Proof for Theorem 1.2.5

Proof of Theorem 1.2.5. Let $z \in X$ and $R > 0$ and set $B = B(z, R)$. Let φ be given as in Theorem 1.2.5. Set

$$t := (1 + d_B(\vartheta, \theta))^{-\frac{\tau}{2+a}}$$

and define $\tilde{\varphi}$ as in Lemma 1.8.1. Let ϑ and θ be in Θ. Then

$$\left| \int e(\vartheta(x) - \theta(x))\varphi(x) \, \mathrm{d}\mu(x) \right|$$

$$\leq \left| \int e(\vartheta(x) - \theta(x))\tilde{\varphi}(x) \, \mathrm{d}\mu(x) \right|$$

$$+ \left| \int e(\vartheta(x) - \theta(x))(\varphi(x) - \tilde{\varphi}(x)) \, \mathrm{d}\mu(x) \right|.$$

Using the cancellative condition `eq-vdc-cond` of Θ on the ball $B(z, 2R)$, the term `eql61` is bounded above by

$$2^a \mu(B(z, 2R)) \|\tilde{\varphi}\|_{\mathrm{Lip}(B(z,2R))} (1 + d_{B(z,2R)}(\vartheta, \theta))^{-\tau}.$$

Using the doubling condition `doublingx`, the inequality `eq-secondt`, and the estimate $d_B \le d_{B(z,2R)}$ from the definition, we estimate `eql63` from above by

$$2^{6a} t^{-1-a} \mu(B) \|\varphi\|_{C^\tau(B)} (1 + d_B(\vartheta, \theta))^{-\tau}.$$

The term `eql62` we estimate using `eq-firstt` and that ϑ and θ are real and thus $e(\vartheta)$ and $e(\theta)$ bounded in absolute value by 1. We obtain for `eql62` with `doublingx` the upper bound

$$2^a \mu(B) t^\tau \|\varphi\|_{C^\tau(B)}.$$

Using the definition `eql69` of t and adding `eql64` and `eql65` estimates `eql60` from above by

$$2^{6a} \mu(B) \|\varphi\|_{C^\tau(B)} (1 + d_B(\vartheta, \theta))^{-\frac{\tau}{2+a}}$$

$$+ 2^a \mu(B) \|\varphi\|_{C^\tau(B)} (1 + d_B(\vartheta, \theta))^{-\frac{\tau^2}{2+a}}.$$

This is

$$\le 2^{1+6a} \mu(B) \|\varphi\|_{C^\tau(B)} (1 + d_B(\vartheta, \theta))^{-\frac{\tau^2}{2+a}},$$

where we used $\tau \le 1$. This completes the proof of Theorem 1.2.5.

1.9 Proof of Vitali covering and Hardy–Littlewood

We begin with a classical representation of the Lebesgue norm.

Lemma 1.9.1

Let $1 \le p < \infty$. Then for any measurable function $u : X \to [0, \infty)$ on the measure space X relative to the measure μ we have

$$\|u\|_p^p = p \int_0^\infty \lambda^{p-1} \mu(\{x : u(x) \ge \lambda\}) \, d\lambda.$$

Proof for Lemma 1.9.1

Proof. The left-hand side of `eq-layercake` is by definition

$$\int_X u(x)^p \, d\mu(x).$$

Writing $u(x)$ as an elementary integral in λ and then using Fubini, we write for the last display

$$= \int_X \int_0^{u(x)} p\lambda^{p-1} d\lambda \, d\mu(x)$$

$$= p \int_0^\infty \lambda^{p-1} \mu(\{x : u(x) \geq \lambda\}) d\lambda.$$

This proves the lemma.

The following lemma will be used to define M in the proof of Theorem 1.2.6.

Lemma 1.9.2

For each $r > 0$, there exists a countable collection $C(r) \subset X$ of points such that

$$X \subset \bigcup_{c \in C(r)} B(c, r).$$

Proof for Lemma 1.9.2

Proof. It clearly suffices to construct finite collections $C(r,k)$ such that

$$B(o, r2^k) \subset \bigcup_{c \in C(r,k)} B(c, r),$$

since then the collection $C(r) = \bigcup_{k \in \mathbb{N}} C(r,k)$ has the desired property.Suppose that $Y \subset B(o, r2^k)$ is a collection of points such that for all $y, y' \in Y$ with $y \neq y'$, we have $\rho(y, y') \geq r$. Then the balls $B(y, r/2)$ are pairwise disjoint and contained in $B(o, r2^{k+1})$. If $y \in B(o, r)$, then $B(o, r2^{k+1}) \subset B(y, r2^{k+2})$. Thus, by the doubling property `doublingx`,

$$\mu(B(y, \frac{r}{2})) \geq 2^{-(k+2)a} \mu(B(o, r2^{k+1})).$$

Thus, we have

$$\mu(B(o, r2^{k+1})) \geq \sum_{y \in Y} \mu(B(y, \frac{r}{2})) \geq |Y| 2^{-(k+2)a} \mu(B(o, r2^{k+1})).$$

We conclude that $|Y| \leq 2^{(k+2)a}$. In particular, there exists a set Y of maximal cardinality. Define $C(r,k)$ to be such a set.If $x \in B(o, r2^k)$ and $x \notin C(r,k)$, then there must exist $y \in C(r,k)$ with $\rho(x, y) < r$. Thus $C(r,k)$ has the desired property.

We turn to the proof of Theorem 1.2.6.

Proof for Theorem 1.2.6

Proof of Theorem 1.2.6. Let the collection $\mathcal{B}$ be given. We first show `eq-besico`.We recursively choose a finite sequence $B_i \in \mathcal{B}$ for $i \geq 0$ as follows. Assume $B_{i'}$ is already chosen for $0 \leq i' < i$. If there exists a ball $B_i \in \mathcal{B}$ so that B_i is disjoint from all $B_{i'}$ with $0 \leq i' < i$, then choose such a ball $B_i = B(x_i, r_i)$ with maximal r_i.If there is no such ball, stop the selection and set $i'' := i$.By disjointedness of the chosen balls and since $0 \leq u$, we have

$$\sum_{0 \leq i < i''} \int_{B_i} u(x)\, d\mu(x) \leq \int_X u(x)\, d\mu(x).$$

By `eq-ball-assumption`, we conclude

$$\lambda \sum_{0 \leq i < i''} \mu(B_i) \leq \int_X u(x)\, d\mu(x).$$

Let $x \in \bigcup \mathcal{B}$. Choose a ball $B' = B(x', r') \in \mathcal{B}$ such that $x \in B'$. If B' is one of the selected balls, then

$$x \in \bigcup_{0 \leq i < i''} B_i \subset \bigcup_{0 \leq i < i''} B(x_i, 3r_i).$$

If B' is not one of the selected balls, then as it is not selected at time i'', there is a selected ball B_i with $B' \cap B_i \neq \emptyset$. Choose such B_i with minimal index i. As B' is therefore disjoint from all balls $B_{i'}$ with $i' < i$ and as it was not selected in place of B_i, we have $r_i \geq r'$.Using a point y in the intersection of B_i and B', we conclude by the triangle inequality

$$\rho(x_i, x') \leq \rho(x_i, y) + \rho(x', y) \leq r_i + r' \leq 2r_i.$$

By the triangle inequality again, we further conclude

$$\rho(x_i, x) \leq \rho(x_i, x') + \rho(x', x) \leq 2r_i + r' \leq 3r_i.$$

It follows that

$$x \in \bigcup_{0 \leq i < i''} B(x_i, 3r_i).$$

With `3rone` and `3rtwo`, we conclude

$$\bigcup \mathcal{B} \subset \bigcup_{0 \leq i < i''} B(x_i, 3r_i).$$

With the doubling property `doublingx` applied twice, we conclude

$$\mu(\bigcup \mathcal{B}) \leq \sum_{0 \leq i < i''} \mu(B(x_i, 3r_i)) \leq 2^{2a} \sum_{0 \leq i < i''} \mu(B_i).$$

With eqbes1 and eqbes2 we conclude eq-besico.We turn to the proof of eq-hlm. We first consider the case $p_1 = 1$ and recall $M_{\mathcal{B}} = M_{\mathcal{B},1}$. We write for the p_2-th power of left-hand side of eq-hlm with Lemma 1.9.1 and a change of variables

$$\|M_{\mathcal{B}}u(x)\|_{p_2}^{p_2} = p_2 \int_0^\infty \lambda^{p_2-1}\mu(\{x : M_{\mathcal{B}}u(x) \geq \lambda\})d\lambda$$

$$= 2^{p_2}p_2 \int_0^\infty \lambda^{p_2-1}\mu(\{x : M_{\mathcal{B}}u(x) \geq 2\lambda\})d\lambda.$$

Using Lemma 1.9.1 and a change of variables, fix $\lambda \geq 0$ and let $x \in X$ satisfy $M_{\mathcal{B}}u(x) \geq 2\lambda$. By definition of $M_{\mathcal{B}}$, there is a ball $B' \in \mathcal{B}$ such that $x \in B'$ and

$$\int_{B'} u(y)\, d\mu(y) \geq 2\lambda\mu(B').$$

Define $u_\lambda(y) := 0$ if $|u(y)| < \lambda$ and $u_\lambda(y) := u(y)$ if $|u(y)| \geq \lambda$. Then with eqbesi10

$$\int_{B'} u_\lambda(y)\, d\mu(y) = \int_{B'} u(y)\, d\mu(y) - \int_{B'} (u - u_\lambda)(y)d\mu(y)$$

$$\geq 2\lambda\mu(B') - \int_{B'} (u - u_\lambda)(y)d\mu(y).$$

As $(u - u_\lambda)(y) \leq \lambda$ by definition, we can estimate the last display by

$$\geq 2\lambda\mu(B') - \int_{B'} \lambda\, d\mu(y) = \lambda\mu(B').$$

Hence x is contained in $\bigcup(\mathcal{B}_\lambda)$, where $\mathcal{B}_\lambda$ is the collection of balls B'' in $\mathcal{B}$ such that

$$\int_{B''} u_\lambda(y)\, d\mu(y) \geq \lambda\mu(B'').$$

We have thus seen

$$\{x : M_{\mathcal{B}}u(x) \geq 2\lambda\} \subset \bigcup \mathcal{B}_\lambda.$$

Applying eq-besico to the collection $\mathcal{B}_\lambda$ gives

$$\lambda\mu(\{x : M_{\mathcal{B}}u(x) \geq 2\lambda\}) \leq 2^{2a} \int u_\lambda(x)\, dx.$$

With Lemma 1.9.1,

$$\lambda\mu(\{x : M_{\mathcal{B}}u(x) \geq 2\lambda\}) \leq 2^{2a} \int_0^\infty \mu(\{x : |u_\lambda(x)| \geq \lambda'\})\, d\lambda'.$$

By definition of u_λ, making a case distinction between $\lambda \geq \lambda'$ and $\lambda < \lambda'$, we see that

$$\mu(\{x : |u_\lambda(x)| \geq \lambda'\}) \leq \mu(\{x : |u(x)| \geq \max(\lambda, \lambda')\}).$$

We obtain with eqbesil1, eqbesil2, and eqbesil3

$$\|M_{\mathcal{B}} u(x)\|_{p_2}^{p_2}$$

$$\leq 2^{p_2+2a} p_2 \int_0^\infty \lambda^{p_2-2} \int_0^\infty \mu(\{x : |u(x)| \geq \max(\lambda, \lambda')\})\, d\lambda' d\lambda.$$

We split the integral into $\lambda \geq \lambda'$ and $\lambda < \lambda'$ and resolve the maximum correspondingly.We have for $\lambda \geq \lambda'$ with Lemma 1.9.1

$$\int_0^\infty \lambda^{p_2-2} \int_0^\lambda \mu(\{x : |u(x)| \geq \lambda\})\, d\lambda' d\lambda$$

$$= \int_0^\infty \lambda^{p_2-1} \mu(\{x : |u(x)| \geq \lambda\}) d\lambda$$

$$= p_2^{-1} \|u\|_{p_2}^{p_2}.$$

We have for $\lambda < \lambda'$ with Fubini and Lemma 1.9.1

$$\int_0^\infty \lambda^{p_2-2} \int_\lambda^\infty \mu(\{x : |u(x)| \geq \lambda'\})\, d\lambda' d\lambda$$

$$= \int_0^\infty \int_0^{\lambda'} \lambda^{p_2-2} \mu(\{x : |u(x)| \geq \lambda'\}) d\lambda d\lambda'$$

$$= (p_2 - 1)^{-1} \int_0^\infty (\lambda')^{p_2-1} \mu(\{x : |u(x)| \geq \lambda'\}) d\lambda'$$

$$= (p_2 - 1)^{-1} p_2^{-1} \|u\|_{p_2}^{p_2}.$$

Adding the two estimates eqbesil4 and eqbesil5 gives

$$\|M_{\mathcal{B}} u(x)\|_{p_2}^{p_2} \leq 2^{p_2+2a}(1+(p_2-1)^{-1}) \|u\|_{p_2}^{p_2} = 2^{p_2+2a} p_2 (p_2-1)^{-1} \|u\|_{p_2}^{p_2}.$$

With $a \geq 1$ and $p_2 > 1$, taking the p_2-th root, we obtain eq-hlm. We turn to the case of general $1 \leq p_1 < p_2$. We have

$$M_{\mathcal{B},p_1} u = (M_{\mathcal{B}}(|u|^{p_1}))^{\frac{1}{p_1}}.$$

Applying the special case of eq-hlm for $M_{\mathcal{B}}$ gives

$$\|M_{\mathcal{B},p_1} u\|_{p_2} = \|M_{\mathcal{B}}(|u|^{p_1})\|_{p_2/p_1}^{\frac{1}{p_1}}$$

$$\leq 2^{2a}(p_2/p_1)(p_2/p_1 - 1)^{-1} \|(|u|^{p_1})\|_{p_2/p_1}^{\frac{1}{p_1}} = 2^{2a} p_2 (p_2 - p_1)^{-1} \|u\|_{p_2}.$$

This proves `eq-hlm` in general.Now we construct the operator M satisfying `eq-ball-av` and `eq-hlm-2`. For each $k \in \mathbb{Z}$ we choose a countable set $C(2^k)$ as in Lemma 1.9.2. Define

$$\mathcal{B}_\infty = \{B(c, 2^k) \ : \ c \in C(2^k), k \in \mathbb{Z}\}.$$

By Lemma 1.9.2, this is a countable collection of balls. We choose an enumeration $\mathcal{B}_\infty = \{B_1, \dots\}$ and define

$$\mathcal{B}_n = \{B_1, \dots, B_n\}.$$

We define

$$Mw := 2^{2a} \sup_{n \in \mathbb{N}} M_{\mathcal{B}_n} w.$$

This function is measurable for each measurable w, since it is a countable supremum of measurable functions. Estimate `eq-hlm-2` follows immediately from `eq-hlm` and the monotone convergence theorem.It remains to show `eq-ball-av`. Let $B = B(x, r) \subset X$. Let k be the smallest integer such that $2^k \geq r$, in particular we have $2^k < 2r$. By definition of $C(2^k)$, there exists $c \in C(2^k)$ with $x \in B(c, 2^k)$. By the triangle inequality, we have $B(c, 2^k) \subset B(x, 4r)$, and hence by the doubling property `doublingx`

$$\mu(B(c, 2^k)) \leq 2^{2a} \mu(B(x, r)).$$

It follows that for each $z \in B(x, r)$

$$\begin{aligned} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |w(y)| \, \mathrm{d}\mu(y) &\leq \frac{2^{2a}}{\mu(B(c, 2^k))} \int_{B(c,2^k)} |w(y)| \, \mathrm{d}\mu(y) \\ &\leq Mw(z). \end{aligned}$$

This completes the proof.

1.10 Two-sided Metric Space Carleson

We prove a variant of Theorem 1.1.1.1 for a two-sided Calderon–Zygmund kernel on the doubling metric measure space (X, ρ, μ, a). This means a one-sided Calderon–Zygmund kernel K which additionally satisfies, for all $x, x', y \in X$ with $x \neq y$ and $2\rho(x, x') \leq \rho(x, y)$,

$$|K(x, y) - K(x', y)| \leq \left(\frac{\rho(x, x')}{\rho(x, y)} \right)^{\frac{1}{a}} \frac{2^{a^3}}{V(x, y)}.$$

By the additional regularity, we can weaken the assumption `nontanbound` to a family of operators that is easier to work with in applications. Namely, for

$r > 0$, $x \in X$, and a bounded measurable function $f : X \to \mathbb{C}$ supported on a set of finite measure, define

$$T_r f(x) := \int_{r \le \rho(x,y)} K(x,y) f(y)\, d\mu(y) = \int_{X \setminus B(x,r)} K(x,y) f(y)\, d\mu(y).$$

Theorem 1.10.1

For all integers $a \ge 4$ and real numbers $1 < q \le 2$, the following holds. Let (X, ρ, μ, a) be a doubling metric measure space. Let Θ be a cancellative compatible collection of functions and let K be a two-sided Calderon–Zygmund kernel on (X, ρ, μ, a). Assume that for every bounded measurable function g on X supported on a set of finite measure and all $r > 0$ we have

$$\|T_r g\|_2 \le 2^{a^3} \|g\|_2.$$

Then for all Borel sets F and G in X and all Borel functions $f : X \to \mathbb{C}$ with $|f| \le \mathbf{1}_F$, we have, with T defined in `def-main-op`,

$$\left| \int_G T f\, d\mu \right| \le \frac{2^{474a^3}}{(q-1)^6} \mu(G)^{1-\frac{1}{q}} \mu(F)^{\frac{1}{q}}.$$

1.11 Proof of The Classical Carleson Theorem

The convergence of partial Fourier sums is proved in section 1.11 (p. 126) in two steps. In the first step, we establish convergence on a suitable dense subclass of functions. We choose smooth functions as subclass; the convergence is stated in Lemma 1.11.1.2 and proved in section 1.11 (p. 132). In the second step, one controls the relevant error of approximating a general function by a function in the subclass. This is stated in Lemma 1.11.1.3 and proved in section 1.11 (p. 140).

The proof relies on a bound on the real Carleson maximal operator stated in Lemma 1.11.1.5 and proved in section 1.11 (p. 144), which involves showing that the real line fits into the setting of section 1.2 (p. 8). This latter proof refers to the two-sided variant of the Carleson theorem, Theorem 1.10.1. Two assumptions in Theorem 1.1.1.1 require more work. The boundedness of the operator T_r defined earlier is established in Lemma 1.11.1.6; this lemma is proved in section 1.11 (p. 133). The cancellative property is verified by Lemma 1.11.1.7, which is proved in section 1.11 (p. 139). Several further auxiliary lemmas are stated and proved in section 1.11 (p. 126); the proof of one of these auxiliary lemmas, Lemma 1.11.1.10, is done in section 1.11 (p. 140). All subsections past section 1.11 (p. 126) are mutually independent.

The classical Carleson theorem

Let a uniformly continuous 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$ and $\epsilon > 0$ be given. Let

$$C_{a,q} := \frac{2^{474a^3}}{(q-1)^6}$$

denote the constant from Theorem 1.10.1. Define

$$\epsilon' := \frac{\epsilon}{4C_\epsilon},$$

where

$$C_\epsilon = \left(\frac{8}{\pi\epsilon}\right)^{\frac{1}{2}} C_{4,2} + \pi.$$

Since f is continuous and periodic, f is uniformly continuous. Thus, there is a $0 < \delta < \pi$ such that for all $x, x' \in \mathbb{R}$ with $|x - x'| \leq \delta$ we have

$$|f(x) - f(x')| \leq \epsilon'.$$

Define

$$f_0 := f * \phi_\delta,$$

where ϕ_δ is a nonnegative smooth bump function with $\operatorname{supp}(\phi_\delta) \subset (-\delta, \delta)$ and $\int_{\mathbb{R}} \phi_\delta(x)\, dx = 1$.

Lemma 1.11.1.1

The function f_0 is 2π-periodic. The function f_0 is smooth and therefore measurable. The function f_0 satisfies, for all $x \in \mathbb{R}$,

$$|f(x) - f_0(x)| \leq \epsilon'.$$

Proof for Lemma 1.11.1.1

Periodicity follows directly from the definitions. The other properties are part of the Lean library.

We prove this in section 1.11 (p. 132):

Lemma 1.11.1.2

There exists some $N_0 \in \mathbb{N}$ such that for all $N > N_0$ and $x \in [0, 2\pi]$ we have

$$|S_N f_0(x) - f_0(x)| \leq \frac{\epsilon}{4}.$$

We prove this in section 1.11 (p. 140):

Lemma 1.11.1.3

There is a set $E \subset \mathbb{R}$ with Lebesgue measure $|E| \leq \epsilon$ such that for all $x \in [0, 2\pi) \setminus E$ we have

$$\sup_{N \geq 0} |S_N f(x) - S_N f_0(x)| \leq \frac{\epsilon}{4}.$$

We are now ready to prove Theorem 1.1.1. We first prove a version with explicit exceptional sets.

Theorem 1.11.1.4

Let f be a 2π-periodic complex-valued continuous function on $\mathbb{R}$. For all $\epsilon > 0$, there exists a Borel set $E \subset [0, 2\pi]$ with Lebesgue measure $|E| \leq \epsilon$ and a positive integer N_0 such that for all $x \in [0, 2\pi] \setminus E$ and all integers $N > N_0$, we have

$$|f(x) - S_N f(x)| \leq \epsilon.$$

Proof for Theorem 1.11.1.4

Let N_0 be as in Lemma 1.11.1.2. For every $x \in [0, 2\pi) \setminus E$ and every $N > N_0$ we have by the triangle inequality

$$|f(x) - S_N f(x)| \leq |f(x) - f_0(x)| + |f_0(x) - S_N f_0(x)| + |S_N f_0(x) - S_N f(x)|.$$

Using Lemma 1.11.1.1, Lemma 1.11.1.2, and Lemma 1.11.1.3, the right-hand side is bounded by $\epsilon' + \frac{\epsilon}{4} + \frac{\epsilon}{4} \leq \epsilon$. This shows the desired estimate for the given E and N_0.

Now we turn to the proof of the statement of Carleson theorem given in the introduction.

Proof for Theorem 1.1.1

Proof of Theorem 1.1.1. By applying Theorem 1.11.1.4 with a sequence of $\epsilon_n := 2^{-n}\delta$ for $n \geq 1$ and taking the union of corresponding exceptional sets E_n, we see that outside a set of measure δ, the partial Fourier sums converge pointwise for $N \to \infty$. Applying this with a sequence of δ shrinking to zero and taking the intersection of the corresponding exceptional sets, which has measure zero, we see that the Fourier series converges outside a set of measure zero.

Let $\kappa : \mathbb{R} \to \mathbb{C}$ be defined by $\kappa(0) = 0$, by

$$\kappa(x) = \frac{1 - |x|}{1 - e^{ix}}$$

for $0 < |x| < 1$, and by $\kappa(x) = 0$ for $|x| \geq 1$. This function is continuous at every point x with $|x| > 0$.

The proof of Lemma 1.11.1.3 will use Lemma 1.11.1.5, which is proved in section 1.11 (p. 144) as an application of Theorem 1.1.1.1.

Lemma 1.11.1.5

Let F, G be Borel subsets of $\mathbb{R}$ with finite measure. Let f be a bounded measurable function on $\mathbb{R}$ with $|f| \le \mathbf{1}_F$. Then

$$\left| \int_G Tf(x)\, dx \right| \le C_{4,2} |F|^{\frac{1}{2}} |G|^{\frac{1}{2}},$$

where

$$Tf(x) = \sup_{n\in\mathbb{Z}} \sup_{r>0} \left| \int_{r<|x-y|<1} f(y)\kappa(x-y)e^{iny}\, dy \right|.$$

One of the main assumptions of Theorem 1.10.1, concerning the operator T_r defined earlier, is verified by the following lemma, which is proved in section 1.11 (p. 133).

Lemma 1.11.1.6

Let $0 < r$. Let f be a bounded measurable function on $\mathbb{R}$. Then

$$\|H_r f\|_2 \le 2^9 \|f\|_2,$$

where

$$H_r f(x) := T_r f(x) = \int_{r\le|x-y|} \kappa(x-y) f(y)\, dy.$$

The next lemma will be used to verify that the collection $\mathfrak{A}$ of modulation functions in our application of Theorem 1.1.1.1 satisfies the condition `eq-vdc-cond`. It is proved in section 1.11 (p. 139).

Lemma 1.11.1.7

Let $\alpha \le \beta$ be real numbers. Let $g : \mathbb{R} \to \mathbb{C}$ be a measurable function and assume

$$\|g\|_{\mathrm{Lip}(\alpha,\beta)} := \sup_{\alpha\le x\le\beta} |g(x)| + \frac{|\beta-\alpha|}{2} \sup_{\alpha\le x<y\le\beta} \frac{|g(y)-g(x)|}{|y-x|} < \infty.$$

Then, for every $n \in \mathbb{Z}$,

$$\int_\alpha^\beta g(x)e^{inx}\, dx \le 2\pi|\beta-\alpha| \|g\|_{\mathrm{Lip}(\alpha,\beta)} (1 + |n||\beta-\alpha|)^{-1}.$$

We close this section with five lemmas that are used across the following subsections.

Lemma 1.11.1.8

For every 2π-periodic bounded measurable f and every $N \ge 0$,

$$S_N f(x) = \frac{1}{2\pi} \int_0^{2\pi} f(y) K_N(x-y)\, dy,$$

where K_N is the 2π-periodic continuous function on $\mathbb{R}$ given by

$$\sum_{n=-N}^{N} e^{inx'}.$$

If $e^{ix'} \neq 1$, then

$$K_N(x') = \frac{e^{iNx'}}{1-e^{-ix'}} + \frac{e^{-iNx'}}{1-e^{ix'}}.$$

Proof for Lemma 1.11.1.8

By definitions and interchanging sum and integral,

$$S_N f(x) = \sum_{n=-N}^{N} \widehat{f}_n e^{inx} = \sum_{n=-N}^{N} \frac{1}{2\pi} \int_0^{2\pi} f(y) e^{in(x-y)}\, dy.$$

Thus

$$S_N f(x) = \frac{1}{2\pi} \int_0^{2\pi} f(y) \sum_{n=-N}^{N} e^{in(x-y)}\, dy.$$

This proves the first statement of the lemma. By a telescoping sum, for every $x' \in \mathbb{R}$ we have

$$\left(e^{\frac{1}{2}ix'} - e^{-\frac{1}{2}ix'}\right) \sum_{n=-N}^{N} e^{inx'} = e^{(N+\frac{1}{2})ix'} - e^{-(N+\frac{1}{2})ix'}.$$

If $e^{ix'} \neq 1$, the first factor on the left-hand side is not zero and we may divide by it to obtain

$$\sum_{n=-N}^{N} e^{inx'} = \frac{e^{i(N+\frac{1}{2})x'}}{e^{\frac{1}{2}ix'} - e^{-\frac{1}{2}ix'}} - \frac{e^{-i(N+\frac{1}{2})x'}}{e^{\frac{1}{2}ix'} - e^{-\frac{1}{2}ix'}} = \frac{e^{iNx'}}{1-e^{-ix'}} + \frac{e^{-iNx'}}{1-e^{ix'}}.$$

This proves the second part of the lemma.

Lemma 1.11.1.9

Let $\eta > 0$ and $-2\pi + \eta \leq x \leq 2\pi - \eta$ with $|x| \geq \eta$. Then

$$|1 - e^{ix}| \geq \frac{2}{\pi}\eta.$$

Proof for Lemma 1.11.1.9

We have

$$|1 - e^{ix}| = \sqrt{(1 - \cos x)^2 + \sin^2 x} \geq |\sin x|.$$

For $0 \leq x \leq \pi/2$, concavity of sin on $[0, \pi]$, together with $\sin(0) = 0$ and $\sin(\pi/2) = 1$, gives

$$|\sin x| \geq \frac{2}{\pi} x \geq \frac{2}{\pi} \eta.$$

The remaining intervals $x \in \frac{m\pi}{2} + [0, \frac{\pi}{2}]$ for $m \in \{-4, -3, -2, -1, 1, 2, 3\}$ are handled similarly.

The following lemma will be proved in section 1.11 (p. 140).

Lemma 1.11.1.10

Let f be a bounded 2π-periodic measurable function. Then, for all $N \geq 0$,

$$\|S_N f\|_{L^2[0,2\pi]} \leq \|f\|_{L^2[0,2\pi]}.$$

Lemma 1.11.1.11

For $x, y \in \mathbb{R}$ with $x \neq y$,

$$|\kappa(x - y)| \leq 2^2 (2|x - y|)^{-1}.$$

Proof for Lemma 1.11.1.11

Fix $x \neq y$. If $\kappa(x - y) = 0$, the estimate is immediate. Otherwise $0 < |x - y| < 1$, and

$$|\kappa(x - y)| = \left| \frac{1 - |x - y|}{1 - e^{i(x-y)}} \right|.$$

Using Lemma 1.11.1.9,

$$|\kappa(x - y)| \leq \frac{1}{|1 - e^{i(x-y)}|} \leq \frac{2}{|x - y|},$$

which proves the stated bound.

Lemma 1.11.1.12

For $x, y, y' \in \mathbb{R}$ with $x \neq y, y'$ and

$$2|y - y'| \leq |x - y|,$$

we have

$$|\kappa(x - y) - \kappa(x - y')| \leq 2^8 \frac{1}{|x - y|} \frac{|y - y'|}{|x - y|}.$$

Proof for Lemma 1.11.1.12

Upon replacing y by $y-x$ and y' by $y'-x$ in the left-hand side of the closeness assumption, we can assume that $x=0$. Then the assumption implies that y and y' have the same sign. Since $\kappa(y)=\bar{\kappa}(-y)$, we can assume that they are both positive, and then it follows that

$$\frac{y}{2} \le y'.$$

We distinguish four cases.If $y, y' \le 1$, then

$$|\kappa(-y)-\kappa(-y')| = \left|\frac{1-y}{1-e^{-iy}} - \frac{1-y'}{1-e^{-iy'}}\right|$$

and by the fundamental theorem of calculus this equals

$$\left|\int_{y'}^{y} \frac{-1+e^{-it}+i(1-t)e^{it}}{(1-e^{-it})^2}\, dt\right|.$$

Using $y' \ge y/2$ and Lemma 1.11.1.9, we bound this by

$$\le |y-y'| \sup_{\frac{y}{2}\le t\le 1} \frac{3}{|1-e^{-it}|^2} \le 3|y-y'| \left(2\frac{2}{y}\right)^2 \le 2^6 \frac{|y-y'|}{|y|^2}.$$

If $y \le 1$ and $y' > 1$, then $\kappa(-y')=0$ and the first case gives

$$|\kappa(-y)-\kappa(-y')| = |\kappa(-y)-\kappa(-1)| \le 2^6 \frac{|y-1|}{|y|^2} \le 2^6 \frac{|y-y'|}{|y|^2}.$$

Similarly, if $y > 1$ and $y' \le 1$, then $\kappa(-y)=0$ and the first case gives

$$|\kappa(-y)-\kappa(-y')| = |\kappa(-y')-\kappa(-1)| \le 2^6 \frac{|y'-1|}{|y'|^2} \le 2^6 \frac{|y-y'|}{|y'|^2}.$$

Using again $y' \ge y/2$, this is bounded by

$$\le 2^6 \frac{|y-y'|}{|y/2|^2} = 2^8 \frac{|y-y'|}{|y|^2}.$$

Finally, if $y, y' > 1$, then

$$|\kappa(-y)-\kappa(-y')| = 0 \le 2^8 \frac{|y-y'|}{|y|^2}.$$

Smooth functions

Lemma 1.11.2.1

Let $f : \mathbb{R} \to \mathbb{C}$ be 2π-periodic and continuously differentiable, and let $n \in \mathbb{Z} \setminus \{0\}$. Then
$$\widehat{f}_n = \frac{1}{in}\widehat{f'}_n.$$

Proof for Lemma 1.11.2.1

This is part of the Lean library.

Lemma 1.11.2.2

Let $f : \mathbb{R} \to \mathbb{C}$ satisfy
$$\sum_{n\in\mathbb{Z}} |\widehat{f}_n| < \infty.$$
Then
$$\sup_{x\in[0,2\pi]} |f(x) - S_N f(x)| \to 0$$
as $N \to \infty$.

Proof for Lemma 1.11.2.2

This is part of the Lean library.

Lemma 1.11.2.3

Let $f : \mathbb{R} \to \mathbb{C}$ be 2π-periodic and twice continuously differentiable. Then
$$\sup_{x\in[0,2\pi]} |f(x) - S_N f(x)| \to 0$$
as $N \to \infty$.

Proof for Lemma 1.11.2.3

By Lemma 1.11.2.2, it suffices to show that the Fourier coefficients $\widehat{f}_n$ are summable. Applying Lemma 1.11.2.1 twice and using the fact that f'' is continuous and therefore bounded on $[0, 2\pi]$, we compute
$$\begin{aligned}\sum_{n\in\mathbb{Z}} |\widehat{f}_n| &= |\widehat{f}_0| + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^2}|\widehat{f''}_n| \\ &\le |\widehat{f}_0| + \left(\sup_{x\in[0,2\pi]} |f(x)|\right) \cdot \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^2} < \infty.\end{aligned}$$

Proof for Lemma 1.11.1.2

Lemma 1.11.1.2 now follows directly from Lemma 1.11.2.3.

The truncated Hilbert transform

Let M_n be the modulation operator on measurable 2π-periodic functions defined by

$$M_n g(x) = g(x)e^{inx}.$$

Define the approximate Hilbert transform by

$$L_N g = \frac{1}{N}\sum_{n=0}^{N-1} M_{n+N} S_{N+n} M_{-N-n} g.$$

Lemma 1.11.3.1

For every bounded measurable 2π-periodic function g,

$$\|L_N g\|_{L^2[0,2\pi]} \leq \|g\|_{L^2[0,2\pi]}.$$

Proof for Lemma 1.11.3.1

We have

$$\|M_n g\|^2_{L^2[0,2\pi]} = \int_0^{2\pi} |e^{inx} g(x)|^2 \, dx = \int_0^{2\pi} |g(x)|^2 \, dx = \|g\|^2_{L^2[0,2\pi]}.$$

By the triangle inequality, the square root of this identity, and Lemma 1.11.1.10,

$$\|L_N g\|_{L^2[0,2\pi]} = \left\| \frac{1}{N}\sum_{n=0}^{N-1} M_{n+N} S_{N+n} M_{-N-n} g \right\|_{L^2[0,2\pi]}$$

$$\leq \frac{1}{N}\sum_{n=0}^{N-1} \|M_{n+N} S_{N+n} M_{-N-n} g\|_{L^2[0,2\pi]}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} \|S_{N+n} M_{-N-n} g\|_{L^2[0,2\pi]} \leq \frac{1}{N}\sum_{n=0}^{N-1} \|M_{-N-n} g\|_{L^2[0,2\pi]} = \|g\|_{L^2[0,2\pi]}.$$

Lemma 1.11.3.2

Let f be a bounded 2π-periodic function. For any $0 \leq x \leq 2\pi$,

$$\int_0^{2\pi} f(y)\, dy = \int_{-x}^{2\pi - x} f(y)\, dy = \int_{-\pi}^{\pi} f(y-x)\, dy.$$

Proof for Lemma 1.11.3.2

By periodicity and change of variables,

$$\int_{-x}^{0} f(y)\,dy = \int_{-x}^{0} f(y+2\pi)\,dy = \int_{2\pi-x}^{2\pi} f(y)\,dy.$$

Breaking up the domain of integration and using this identity gives

$$\int_{0}^{2\pi} f(y)\,dy = \int_{0}^{2\pi-x} f(y)\,dy + \int_{2\pi-x}^{2\pi} f(y)\,dy$$

$$= \int_{0}^{2\pi-x} f(y)\,dy + \int_{-x}^{0} f(y)\,dy = \int_{-x}^{2\pi-x} f(y)\,dy.$$

The second identity follows by substituting y by $y-x$.

Lemma 1.11.3.3

Let f and g be bounded nonnegative measurable 2π-periodic functions on $\mathbb{R}$. Then

$$\left(\int_{0}^{2\pi}\left(\int_{0}^{2\pi} f(y)g(x-y)\,dy\right)^{2} dx\right)^{\frac{1}{2}} \le \|f\|_{L^2[0,2\pi]}\|g\|_{L^1[0,2\pi]}.$$

Proof for Lemma 1.11.3.3

Using Fubini and Lemma 1.11.3.2, we have

$$\int_{0}^{2\pi}\int_{0}^{2\pi} f(y)^2 g(x-y)\,dy\,dx = \int_{0}^{2\pi} f(y)^2 \int_{0}^{2\pi} g(x-y)\,dx\,dy$$

$$= \int_{0}^{2\pi} f(y)^2 \int_{0}^{2\pi} g(x)\,dx\,dy = \|f\|_{L^2[0,2\pi]}^2\|g\|_{L^1[0,2\pi]}.$$

Let h be the nonnegative square root of g; then h is bounded and 2π-periodic with $h^2 = g$. By Cauchy–Schwarz and the preceding identity, the square of the left-hand side is at most

$$\int_{0}^{2\pi}\left(\int_{0}^{2\pi} f(y)^2 g(x-y)\,dy\right)\left(\int_{0}^{2\pi} g(x-y)\,dy\right) dx$$

$$= \|f\|_{L^2[0,2\pi]}^2\|g\|_{L^1[0,2\pi]}^2.$$

Taking square roots proves the lemma.

For $0 < r < 1$, define the kernel k_r to be the 2π-periodic function

$$k_r(x) := \min\left(r^{-1}, 1 + \frac{r}{|1-e^{ix}|^2}\right),$$

where the minimum is understood to be r^{-1} when $1 = e^{ix}$.

Lemma 1.11.3.4

Let g, f be bounded measurable 2π-periodic functions. Let $0 < r < \pi$. Assume $|g(x)| \le k_r(x)$ for all x. Let

$$h(x) = \int_0^{2\pi} f(y)g(x-y)\, dy.$$

Then

$$\|h\|_{L^2[0,2\pi]} \le 17\|f\|_{L^2[-\pi,\pi]}.$$

Proof for Lemma 1.11.3.4

By monotonicity of the integral and the pointwise bound on g,

$$\|g\|_{L^1[0,2\pi]} \le \int_0^{2\pi} k_r(x)\, dx.$$

Using the symmetry $k_r(x) = k_r(-x)$, the assumption, and Lemma 1.11.1.9, the last display is

$$\begin{aligned} &2\int_0^{\pi} \min\left(\frac{1}{r}, 1 + \frac{r}{|1-e^{ix}|^2}\right) dx \\ &\le 2\int_0^{r} \frac{1}{r}\, dx + 2\int_r^{\pi} \left(1 + \frac{64r}{x^2}\right) dx \\ &\le 2 + 2\pi + 2\left(\frac{4r}{r} - \frac{4r}{\pi}\right) \le 17. \end{aligned}$$

Together with Lemma 1.11.3.3, this proves the lemma.

Lemma 1.11.3.5

Let $0 < r < 1$, and let N be the smallest integer larger than $1/r$. There is a 2π-periodic continuous function L' on $\mathbb{R}$ such that, for all $0 \le x \le 2\pi$ and all 2π-periodic bounded measurable functions f,

$$L_N f(x) = \frac{1}{2\pi}\int_0^{2\pi} f(y)L'(x-y)\, dy.$$

Moreover, for all $-\pi \le x \le \pi$,

$$\left|L'(x) - \mathbf{1}_{\{y\colon r<|y|<1\}}\kappa(x)\right| \le 12k_r(x).$$

Proof for Lemma 1.11.3.5

By definition and Lemma 1.11.1.8,

$$L_N g(x) = \frac{1}{N}\sum_{n=0}^{N-1}\int_0^{2\pi} e^{-i(N+n)x}K_{N+n}(x-y)e^{i(N+n)y}g(y)\, dy.$$

This has the required convolution form with

$$L'(x) = \frac{1}{N}\sum_{n=0}^{N-1} K_{N+n}(x)e^{-i(N+n)x}.$$

With the exponential-sum formula `eqksumexp` from Lemma 1.11.1.8, we have $|K_N(x)| \le 2N+1$ for every x, and hence

$$|L'(x)| \le \frac{1}{N}\sum_{n=0}^{N-1}(2N+2n+1) \le 4N \le 2^3 r^{-1}.$$

Therefore, for $|x| \in [0,r) \cup (1,\pi]$, we have

$$\left|L'(x) - \mathbf{1}_{\{y\colon r<|y|<1\}}(x)\kappa(x)\right| = |L'(x)| \le 2^3 r^{-1}.$$

This proves `eqdifflhil` for $|x| \in [0,r)$ since $k_r(x) = r^{-1}$ in this case.For $e^{ix} \neq 1$, the Hilbert-form expression `eqksumhil` from Lemma 1.11.1.8 gives

$$\begin{aligned} L'(x) &= \frac{1}{N}\sum_{n=0}^{N-1}\left(\frac{e^{i(N+n)x}}{1-e^{-ix}} + \frac{e^{-i(N+n)x}}{1-e^{ix}}\right)e^{i(N+n)x} \\ &= \frac{1}{N}\sum_{n=0}^{N-1}\left(\frac{e^{i2(N+n)x}}{1-e^{-ix}} + \frac{1}{1-e^{ix}}\right) \end{aligned}$$

and hence

$$L'(x) = \frac{1}{1-e^{ix}} + \frac{1}{N}\frac{e^{i2Nx}}{1-e^{-ix}}\sum_{n=0}^{N-1} e^{i2nx}.$$

Therefore

$$L'(x) - \mathbf{1}_{\{y\colon r<|y|<1\}}\kappa(x) = L''(x) + \frac{1-\mathbf{1}_{\{y\colon r<|y|<1\}}(x)(1-|x|)}{1-e^{ix}},$$

where

$$L''(x) := \frac{1}{N}\frac{e^{i2Nx}}{1-e^{-ix}}\sum_{n=0}^{N-1} e^{i2nx}.$$

For $x \in [-\pi, r] \cup [r,\pi]$, using Lemma 1.11.1.9 gives

$$\left|\frac{1-\mathbf{1}_{\{y\colon r<|y|<1\}}(x)(1-|x|)}{1-e^{ix}}\right| = \left|\frac{\min(|x|,1)}{1-e^{ix}}\right| \le \frac{2\min(|x|,1)}{|x|} \le 2k_r(x).$$

It remains to estimate L''. If the real part of e^{ix} is negative, then $1 \le |1-e^{-ix}| \le 2$, so

$$|L''(x)| \le \frac{1}{N}\sum_{n=0}^{N-1} 1 = 1 \le 1 + \frac{r}{|1-e^{ix}|^2}.$$

If the real part is positive and still $e^{ix} \neq \pm 1$, telescoping gives

$$(1 - e^{2ix}) \sum_{n=0}^{N-1} e^{i2nx} = 1 - e^{i2Nx}.$$

As $e^{2ix} \neq 1$, we may divide by $1 - e^{2ix}$ and insert this into `eqhil3` to obtain

$$L''(x) = \frac{1}{N} \frac{e^{i2Nx}}{1 - e^{-ix}} \frac{1 - e^{i2Nx}}{1 - e^{2ix}},$$

and, using the nonnegativity of the real part of e^{ix},

$$|L''(x)| \leq \frac{2}{N} \frac{1}{|1 - e^{ix}|} \frac{1}{|1 - e^{2ix}|} = \frac{2}{N} \frac{1}{|1 - e^{ix}|^2} \frac{1}{|1 + e^{ix}|} \leq \frac{2r}{|1 - e^{ix}|^2} \leq 2\left(1 + \frac{r}{|1 - e^{ix}|^2}\right).$$

For $|x| \in [r, \pi]$, one checks $1 + \frac{r}{|1-e^{ix}|^2} \leq 5k_r(x)$. Thus the estimates above show that $|L''(x)| \leq 10k_r(x)$ in this range. Together with the previous $2k_r(x)$ bound, this proves `eqdifflhil` for $|x| \in [r, \pi]$. Since $|x| \in [0, r)$ was covered by `eqdiffzero`, this completes the proof.

We now prove Lemma 1.11.1.6.

Proof for Lemma 1.11.1.6

Proof of Lemma 1.11.1.6. We first show that if f is supported in $[1, 4]$, then

$$\|H_r f\|_{L^2[2,3]} \leq 2^8 \|f\|_{L^2(\mathbb{R})}.$$

Let $\tilde{f}$ be the 2π-periodic extension of f to $\mathbb{R}$, and let N be the smallest integer larger than $1/r$. Then the identity `eqdifflhil` shows that the kernels of H_r and $2\pi L_N$ differ by at most $12k_r$ on $[-\pi, \pi]$. Consider $x \in [2, 3]$. When computing $H_r f(x)$ and $2\pi L_N f(x)$, the kernels are computed at points of the form $x - y$ with $f(y) \neq 0$, hence $y \in [1, 4]$. Since $x \in [2, 3]$, the difference $x - y$ is bounded in absolute value by 2 and therefore belongs to $[-\pi, \pi]$, where the above bound holds.Thus for $x \in [2, 3]$,

$$|H_r \tilde{f}(x)| \leq 2\pi |L_N \tilde{f}(x)| + 12 \left| \int_0^{2\pi} k_r(x - y) \tilde{f}(y)\, dy \right|.$$

Taking the L^2 norm and using subadditivity gives

$$\|H_r \tilde{f}\|_{L^2[2,3]} \leq 2\pi \|L_N \tilde{f}\|_{L^2[0,2\pi]} + 12 \left(\int_0^{2\pi} \left| \int_0^{2\pi} k_r(x - y) \tilde{f}(y)\, dy \right|^2 dx \right)^{1/2}.$$

Since κ is supported in $[-1, 1]$, $H_r \tilde{f}$ agrees with $H_r f$ on $[2, 3]$. Using Lemma 1.11.3.1 and Lemma 1.11.3.4, gives

$$\|H_r f\|_{L^2[2,3]} \leq 2\pi \|f\|_{L^2[0,2\pi]} + 12 \cdot 17 \|f\|_{L^2[0,2\pi]},$$

which gives the short-support estimate.Suppose now that f is supported in $[c, c+3]$ for some $c \in \mathbb{R}$. Then $g(x) = f(x-c+1)$ is supported in $[1, 4]$. By a change of variables in `def-H-r`, $H_r g(x) = H_r f(x-c+1)$. Thus, by the short-support estimate,

$$\|H_r f\|_{L^2[c+1,c+2]} \le 2^8 \|f\|_2.$$

Let now f be arbitrary. Since $\kappa(x) = 0$ for $|x| > 1$, for all $x \in [c+1, c+2]$ we have $H_r f(x) = H_r(f\mathbf{1}_{[c,c+3]})(x)$. Thus

$$\int_{c+1}^{c+2} |H_r f(x)|^2 \, dx = \int_{c+1}^{c+2} |H_r(f\mathbf{1}_{[c,c+3]})(x)|^2 \, dx.$$

Applying the short-support bound gives

$$\le 2^{16} \int_c^{c+3} |f(x)|^2 \, dx.$$

Summing over all $c \in \mathbb{Z}$ gives

$$\int_{\mathbb{R}} |H_r f(x)|^2 \, dx \le 3 \cdot 2^{16} \int_{\mathbb{R}} |f(x)|^2 \, dx.$$

This completes the proof.

The proof of the van der Corput Lemma

Proof for Lemma 1.11.1.7

Proof of Lemma 1.11.1.7. Let g be Lipschitz continuous as in the lemma. If $n = 0$, then

$$\int_\alpha^\beta g(x) \, dx \le |\beta-\alpha| \sup_{\alpha \le x \le \beta} |g(x)| \le |\beta-\alpha| \|g\|_{\mathrm{Lip}(\alpha,\beta)} (1+|n||\beta-\alpha|)^{-1}.$$

Assume now $n \neq 0$; by symmetry we may suppose $n > 0$.If $\beta - \alpha < \pi/n$, the triangle inequality gives

$$\left| \int_\alpha^\beta g(x) e^{inx} \, dx \right| \le |\beta-\alpha| \sup_{x \in [\alpha,\beta]} |g(x)| \le 2\pi |\beta-\alpha| \|g\|_{\mathrm{Lip}(\alpha,\beta)} (1+|n||\beta-\alpha|)^{-1}.$$

Now suppose $\pi/n \le \beta - \alpha$. Since $e^{in(x+\pi/n)} = -e^{inx}$, we write

$$\int_\alpha^\beta g(x) e^{inx} \, dx = \frac{1}{2} \int_\alpha^\beta g(x) e^{inx} \, dx - \frac{1}{2} \int_\alpha^\beta g(x) e^{in(x+\pi/n)} \, dx.$$

We split the first integral at $\alpha + \pi/n$ and the second one at $\beta - \pi/n$, and make a change of variables in the second part of the first integral to obtain

$$= \frac{1}{2} \int_\alpha^{\alpha+\pi/n} g(x) e^{inx} \, dx - \frac{1}{2} \int_{\beta-\pi/n}^\beta g(x) e^{in(x+\pi/n)} \, dx$$

$$+\frac{1}{2}\int_{\alpha+\pi/n}^{\beta}\left(g(x)-g\left(x-\frac{\pi}{n}\right)\right)e^{inx}\,dx.$$

The sum of the first two terms is, by the triangle inequality, bounded by

$$\frac{\pi}{n}\sup_{x\in[\alpha,\beta]}|g(x)|.$$

The third term is, by the triangle inequality, at most

$$\frac{1}{2}\int_{\alpha+\pi/n}^{\beta}\left|g(x)-g\left(x-\frac{\pi}{n}\right)\right|\,dx$$

$$\leq \frac{|\beta-\alpha|}{2}\frac{\pi}{n}\sup_{\alpha\leq x<y\leq\beta}\frac{|g(x)-g(y)|}{|x-y|}.$$

Adding the two terms, we obtain

$$\left|\int_{\alpha}^{\beta} g(x)e^{-inx}\,dx\right| \leq \frac{\pi}{n}\|g\|_{\mathrm{Lip}(\alpha,\beta)}.$$

This completes the proof of the lemma, using that with $\frac{\pi}{n}\leq\beta-\alpha$,

$$\frac{\pi}{n}=\frac{2\pi|\beta-\alpha|}{2n|\beta-\alpha|}\leq 2\pi|\beta-\alpha|(1+n|\beta-\alpha|)^{-1},$$

as required.

Partial sums as orthogonal projections

This subsection proves Lemma 1.11.1.10.
Proof for Lemma 1.11.1.10

Proof of Lemma 1.11.1.10. The functions $e_n : x\mapsto e^{inx}$ form an orthonormal basis in $L^2[-\pi,\pi]$, which is already in Mathlib. Therefore

$$\|S_N f\|^2_{L^2[-\pi,\pi]} = \left\|\sum_{n=-N}^{N}\langle\widehat{f}_n, e_n\rangle_{L^2[-\pi,\pi]}\right\|^2_{L^2[-\pi,\pi]}$$

$$=\sum_{n=-N}^{N}|\widehat{f}_n| \leq \sum_{n\in\mathbb{Z}}|\widehat{f}_n| = \|f\|_{L^2[-\pi,\pi]}.$$

This completes the proof.

Difference control

Lemma 1.11.6.1

For all $N \in \mathbb{Z}$ and $x \in [-\pi, \pi] \setminus \{0\}$,

$$\left| K_N(x) - \left(e^{-iNx}\kappa(x) + \overline{e^{-iNx}\kappa(x)} \right) \right| \leq \pi.$$

Proof for Lemma 1.11.6.1

Let $N \in \mathbb{Z}$ and $x \in [-\pi, \pi] \setminus \{0\}$. With Lemma 1.11.1.8, we obtain

$$K_N(x) - \left(e^{-iNx}\kappa(x) + \overline{e^{-iNx}\kappa(x)} \right) = e^{-iNx}\frac{\min(|x|, 1)}{1 - e^{ix}} + e^{iNx}\frac{\min(|x|, 1)}{1 - e^{-ix}}.$$

Using Lemma 1.11.1.9 with $\eta = \min(|x|, 1)$, we bound

$$\left| K_N(x) - \left(e^{-iNx}\kappa(x) + \overline{e^{-iNx}\kappa(x)} \right) \right| \leq \frac{\min(|x|, 1)}{|1 - e^{ix}|} + \frac{\min(|x|, 1)}{|1 - e^{-ix}|} \leq \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Lemma 1.11.6.2

Let $g : \mathbb{R} \to \mathbb{C}$ be a measurable 2π-periodic function such that, for some $\delta > 0$ and every $x \in \mathbb{R}$,

$$|g(x)| \leq \delta.$$

Then for every $x \in [0, 2\pi]$ and $N > 0$,

$$|S_N g(x)| \leq \frac{1}{2\pi}(Tg(x) + T\bar{g}(x)) + \pi\delta.$$

Proof for Lemma 1.11.6.2

Let $x \in [0, 2\pi]$ and $N > 0$. With Lemma 1.11.1.8 we have

$$|S_N g(x)| = \frac{1}{2\pi} \left| \int_0^{2\pi} g(y) K_N(x - y)\, dy \right|.$$

Using the 2π-periodicity of g and K_N, we shift the integration domain and obtain

$$\frac{1}{2\pi} \left| \int_{x-\pi}^{x+\pi} g(y) K_N(x - y)\, dy \right|.$$

By the triangle inequality, this is bounded by

$$\frac{1}{2\pi} \left| \int_{x-\pi}^{x+\pi} g(y) \left(K_N(x - y) - \max(|x - y|, 0) K_N(x - y) \right) \, dy \right|$$

plus

$$\frac{1}{2\pi}\left|\int_{x-\pi}^{x+\pi} g(y)\max(|x-y|,0)K_N(x-y)\,dy\right|.$$

All integrals are well defined, since K_N is bounded by $2N+1$.Using

$$\max(|x-y|,0)K_N(x-y) = e^{-iN(x-y)}\kappa(x-y)+\overline{e^{-iN(x-y)}\kappa(x-y)},$$

Lemma 1.11.6.1, and the bound on g, the integrable difference term is at most $\pi\delta$. By dominated convergence and since $\kappa(x-y)=0$ for $|x-y|>1$, the singular term equals

$$\frac{1}{2\pi}\lim_{r\to 0^+}\left|\int_{r<|x-y|<1} g(y)\max(|x-y|,0)K_N(x-y)\,dy\right|.$$

Bounding the limit by a supremum, rewriting with the preceding identity, and using the triangle inequality gives

$$\leq \frac{1}{2\pi}\sup_{r>0}\left|\int_{r<|x-y|<1} g(y)e^{-iNy}\kappa(x-y)\,dy\right|+\frac{1}{2\pi}\sup_{r>0}\left|\int_{r<|x-y|<1} \overline{g}(y)e^{-iNy}\kappa(x-y)\,dy\right|.$$

By the definition of T, this is bounded by

$$\frac{1}{2\pi}(Tg(x)+T\overline{g}(x)).$$

Lemma 1.11.6.3

Let f be a bounded measurable function on $\mathbb{R}$. Then Tf, as defined earlier, is measurable.

Proof for Lemma 1.11.6.3

Since a countable supremum of measurable functions is measurable, it suffices to show that for every $n\in\mathbb{Z}$,

$$x\mapsto \sup_{r>0}\left|\int_{r<|x-y|<1} f(y)\kappa(x-y)e^{iny}\,dy\right|$$

is measurable. Fix $n\in\mathbb{Z}$. For each $x\in\mathbb{R}$, the map

$$r\mapsto \left|\int_{r<|x-y|<1} f(y)\kappa(x-y)e^{iny}\,dy\right|$$

is continuous on $(0,\infty)$, because the integrand is locally bounded on $0<|x-y|<1$ by the assumptions on f and Lemma 1.11.1.11. Hence, for each $x\in\mathbb{R}$,

$$\sup_{r>0}\left|\int_{r<|x-y|<1} f(y)\kappa(x-y)e^{iny}\,dy\right| = \sup_{r\in\mathbb{Q}_{>0}}\left|\int_{r<|x-y|<1} f(y)\kappa(x-y)e^{iny}\,dy\right|.$$

The right-hand side is again a countable supremum, so it remains to prove that for every $r \in \mathbb{Q}_{>0}$,

$$x \mapsto \left| \int_{r<|x-y|<1} f(y)\kappa(x-y)e^{iny}\, dy \right| = \left| \int \mathbf{1}_{\{r<|x-\cdot|<1\}}(y) f(y)\kappa(x-y)e^{iny}\, dy \right|$$

is measurable. This follows because the integrand is measurable in (x, y).

Lemma 1.11.6.4

Let $g : \mathbb{R} \to \mathbb{C}$ be a measurable 2π-periodic function such that, for some $\delta > 0$ and every $x \in \mathbb{R}$,

$$|g(x)| \le \delta.$$

Then for every $\epsilon > 0$, there exists a measurable set $E \subset [0, 2\pi]$ with $|E| < \epsilon$ such that, for every $x \in [0, 2\pi] \setminus E$ and $N > 0$,

$$|S_N g(x)| \le C_\epsilon \delta,$$

where

$$C_\epsilon = \left(\frac{8}{\pi\epsilon}\right)^{\frac{1}{2}} C_{4,2} + \pi.$$

Proof for Lemma 1.11.6.4

Define

$$E := \left\{ x \in [0, 2\pi] \ : \ \sup_{N>0} |S_N g(x)| > C_\epsilon \delta \right\}.$$

Then the desired bound on $S_N g(x)$ is clear outside E, and it remains to show that $|E| \le \epsilon$. By Lemma 1.11.6.2,

$$E \subset \left\{ x \in [0, 2\pi] \ : \ C_\epsilon \delta < \frac{1}{2\pi}(Tg(x) + T\bar{g}(x)) + \pi\delta \right\} \subset E_1 \cup E_2,$$

where

$$E_1 := \{x \in [0, 2\pi] \ : \ \pi(C_\epsilon - \pi)\delta < Tg(x)\}$$

and

$$E_2 := \{x \in [0, 2\pi] \ : \ \pi(C_\epsilon - \pi)\delta < T\bar{g}(x)\}.$$

By Lemma 1.11.6.3, E_1 and E_2 are measurable. Thus

$$\pi(C_\epsilon - \pi)\delta|E_1| \le \int_{E_1} Tg(x)\, dx = \delta \int_{E_1} T(\delta^{-1} g \mathbf{1}_{[-\pi,3\pi]})(x)\, dx.$$

Applying Lemma 1.11.1.5 with $F = [-\pi, 3\pi]$ and $G = E_1$, this is bounded by

$$\delta \cdot C_{4,2}|F|^{\frac{1}{2}}|E_1|^{\frac{1}{2}} \le (4\pi)^{\frac{1}{2}} C_{4,2}\delta|E_1|^{\frac{1}{2}}.$$

Rearranging gives

$$|E_1| \le \left(\frac{(4\pi)^{\frac{1}{2}} C_{4,2}}{\pi(C_\epsilon - \pi)} \right)^2 = \frac{\epsilon}{2}.$$

The same estimate holds for $|E_2|$. The result follows from $|E| \le |E_1| + |E_2|$.

Proof for Lemma 1.11.1.3

Lemma 1.11.1.3 follows directly from Lemma 1.11.6.4 with $\delta := \epsilon'$.

Carleson on the real line

We prove Lemma 1.11.1.5.

Consider the standard distance function $\rho(x, y) = |x - y|$ on the real line $\mathbb{R}$.

Lemma 1.11.7.1

The space $(\mathbb{R}, \rho)$ is a complete locally compact metric space.

Proof for Lemma 1.11.7.1

This is part of the Lean library.

Lemma 1.11.7.2

For $x \in \mathbb{R}$ and $R > 0$, the ball $B(x, R)$ is the interval $(x - R, x + R)$.

Proof for Lemma 1.11.7.2

Let $x' \in B(x, R)$. By definition, $|x' - x| < R$, hence $x' < x + R$ and $x' > x - R$. Thus $x' \in (x - R, x + R)$. Conversely, if $x' \in (x - R, x + R)$, then $x' - x < R$ and $x - x' < R$, so $|x' - x| < R$ and $x' \in B(x, R)$.

We consider the Lebesgue measure μ on $\mathbb{R}$.

Lemma 1.11.7.3

The measure μ is a sigma-finite nonzero Radon-Borel measure on $\mathbb{R}$.

Proof for Lemma 1.11.7.3

This is part of the Lean library.

Lemma 1.11.7.4

For every $x \in \mathbb{R}$ and $R > 0$,

$$\mu(B(x, R)) = 2R.$$

Proof for Lemma 1.11.7.4

By Lemma 1.11.7.2,

$$\mu(B(x, R)) = \mu((x - R, x + R)) = 2R.$$

Lemma 1.11.7.5

For every $x \in \mathbb{R}$ and $R > 0$,

$$\mu(B(x, 2R)) = 2\mu(B(x, R)).$$

Proof for Lemma 1.11.7.5

By Lemma 1.11.7.4,

$$\mu(B(x, 2R)) = 4R = 2\mu(B(x, R)).$$

The preceding four lemmas show that $(\mathbb{R}, \rho, \mu, 4)$ is a doubling metric measure space. Indeed, we even show that $(\mathbb{R}, \rho, \mu, 1)$ is a doubling metric measure space, but we may relax the estimate in Lemma 1.11.7.5 to conclude that $(\mathbb{R}, \rho, \mu, 4)$ is a doubling metric measure space.

For each $n \in \mathbb{Z}$ define $\mathfrak{a}_n : \mathbb{R} \to \mathbb{R}$ by $\mathfrak{a}_n(x) = nx$. Let $\mathfrak{A}$ be the collection $\{\mathfrak{a}_n : n \in \mathbb{Z}\}$. For every $n \in \mathbb{Z}$ we have $\mathfrak{a}_n(0) = 0$. Define

$$d_{B(x,R)}(\mathfrak{a}_n, \mathfrak{a}_m) := 2R|n - m|.$$

Lemma 1.11.7.6

For every $R > 0$ and $x \in \mathbb{R}$, the function $d_{B(x,R)}$ is a metric on $\mathfrak{A}$.

Proof for Lemma 1.11.7.6

This follows immediately from the fact that the standard metric on $\mathbb{Z}$ is a metric.

Lemma 1.11.7.7

For every $R > 0$ and $x \in \mathbb{R}$, and for all $n, m \in \mathbb{Z}$,

$$\sup_{y,y' \in B(x,R)} |ny - ny' - my + my'| \le 2|n - m|R.$$

Proof for Lemma 1.11.7.7

The right-hand side is the supremum of

$$|(n-m)(y-x)-(n-m)(y'-x)|$$

over $y, y' \in B(x,R)$. The estimate follows from the triangle inequality.

Lemma 1.11.7.8

For any $x, x' \in \mathbb{R}$ and $R, R' > 0$ with $B(x,R) \subset B(x',R')$, and for any $n, m \in \mathbb{Z}$,

$$d_{B(x,R)}(\mathfrak{a}_n, \mathfrak{a}_m) \leq d_{B(x',R')}(\mathfrak{a}_n, \mathfrak{a}_m).$$

Proof for Lemma 1.11.7.8

This follows immediately from the definition of $d_{B(x,R)}$ and $R \leq R'$.

Lemma 1.11.7.9

For any $x, x' \in \mathbb{R}$ and $R > 0$ with $x \in B(x', 2R)$, and any $n, m \in \mathbb{Z}$,

$$d_{B(x',2R)}(\mathfrak{a}_n, \mathfrak{a}_m) \leq 2d_{B(x,R)}(\mathfrak{a}_n, \mathfrak{a}_m).$$

Proof for Lemma 1.11.7.9

With `eqcarl4`, both sides of `firstdbl` are equal to $4R|n-m|$. This proves the lemma.

Lemma 1.11.7.10

For any $x, x' \in \mathbb{R}$ and $R > 0$ with $B(x,R) \subset B(x',2R)$, and any $n, m \in \mathbb{Z}$,

$$2d_{B(x,R)}(\mathfrak{a}_n, \mathfrak{a}_m) \leq d_{B(x',2R)}(\mathfrak{a}_n, \mathfrak{a}_m).$$

Proof for Lemma 1.11.7.10

With `eqcarl4`, both sides of `seconddbl` are equal to $4R|n-m|$. This proves the lemma.

Lemma 1.11.7.11

For every $x \in \mathbb{R}$, $R > 0$, $n \in \mathbb{Z}$, and $R' > 0$, there exist $m_1, m_2, m_3 \in \mathbb{Z}$ such that

$$B' \subset B_1 \cup B_2 \cup B_3,$$

where

$$B' = \{\mathfrak{a} \in \mathfrak{A} : d_{B(x,R)}(\mathfrak{a}, \mathfrak{a}_n) < 2R'\}$$

and, for $j = 1, 2, 3$,

$$B_j = \{\mathfrak{a} \in \mathfrak{A} : d_{B(x,R)}(\mathfrak{a}, \mathfrak{a}_{m_j}) < R'\}.$$

Proof for Lemma 1.11.7.11

Let m_1 be the largest integer smaller than or equal to $n - \frac{R'}{2R}$. Let $m_2 = n$. Let m_3 be the smallest integer larger than or equal to $n + \frac{R'}{2R}$.Let $\mathfrak{a}_{n'} \in B'$. Then, with the definition of $d_{B(x,R)}$, we have

$$2R|n - n'| < 2R'.$$

Assume first $n' \leq m_1$. Then

$$R|m_1 - n'| = R(m_1 - n') = R(m_1 - n) + R(n - n') < -\frac{R'}{2} + R' = \frac{R'}{2},$$

so $\mathfrak{a}_{n'} \in B_1$.Assume next $m_1 < n' < m_3$. Then $\mathfrak{a}_{n'} \in B_2$.Assume finally that $m_3 \leq n'$. Then

$$R|m_3 - n'| = R(n' - m_3) = R(n' - n) + R(n - m_3) < R' - \frac{R'}{2} = \frac{R'}{2},$$

so $\mathfrak{a}_{n'} \in B_3$. This completes the proof.

Lemma 1.11.7.12

For any $x \in \mathbb{R}$ and $R > 0$, and any function $\varphi : X \to \mathbb{C}$ supported on $B' = B(x, R)$ such that

$$\|\varphi\|_{\mathrm{Lip}(B')} = \sup_{x \in B'} |\varphi(x)| + R \sup_{x,y \in B', x \neq y} \frac{|\varphi(x) - \varphi(y)|}{\rho(x,y)}$$

is finite, and for any $n, m \in \mathbb{Z}$,

$$\left| \int_{B'} e(\mathfrak{a}_n(x) - \mathfrak{a}_m(x))\varphi(x)\, d\mu(x) \right| \leq 2\pi\mu(B') \frac{\|\varphi\|_{\mathrm{Lip}(B')}}{1 + d_{B'}(\mathfrak{a}_n, \mathfrak{a}_m)}.$$

Proof for Lemma 1.11.7.12

Set $n' = n - m$. We need to prove

$$\left| \int_{x-R}^{x+R} e^{in'y} \varphi(y)\, dy \right| \leq 4\pi R \|\varphi\|_{\mathrm{Lip}(B')} (1 + 2R|n'|)^{-1}.$$

This follows from Lemma 1.11.1.7 with $\alpha = x - R$ and $\beta = x + R$.

Proof for Lemma 1.11.1.5

The preceding chain of lemmas establishes that $\mathfrak{A}$ is a cancellative compatible collection of functions on $(\mathbb{R}, \rho, \mu, 4)$. Some statements are stronger than needed for $a = 4$, but can be relaxed to the assumptions of Theorem 1.1.1.1 and the desired conclusion.With κ as above, define $K : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ by

$$K(x, y) := \kappa(x - y).$$

The function K is continuous outside the diagonal $x = y$ and vanishes on the diagonal, hence it is measurable.By Lemma 1.11.1.11 and Lemma 1.11.1.12, K is a two-sided Calderon–Zygmund kernel on $(\mathbb{R}, \rho, \mu, 4)$. Lemma 1.11.1.6 verifies the required truncated-operator bound. Thus all assumptions of Theorem 1.10.1 are satisfied, and applying that theorem proves Lemma 1.11.1.5.

Dependency Graph

The dependency graph is available in the HTML output.

Blueprint Summary

The Blueprint summary is available in the HTML output.