Noperthedron

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Contents

1 Introduction

We follow for the most part the structure of Steininger and Yurkevich (2025).

2 The Noperthedron

2.1 Definition of the Noperthedron

We define three points $C_1, C_2, C_3 \in \mathbb{Q}^3$.

$$C_1 := \frac{1}{259375205}\begin{pmatrix}152024884\\0\\210152163\end{pmatrix}, \qquad C_2 := \frac{1}{10^{10}}\begin{pmatrix}6632738028\\6106948881\\3980949609\end{pmatrix},$$

$$C_3 := \frac{1}{10^{10}}\begin{pmatrix}8193990033\\5298215096\\1230614493\end{pmatrix}.$$

Lean code for «def:noperthedron_main»

```
def C1 : Fin 3 → ℚ := (1/259375205) * ![152024884, 0,
↪  210152163]
def C2 : Fin 3 → ℚ := (1/10^10) * ![6632738028, 6106948881,
↪  3980949609]
def C3 : Fin 3 → ℚ := (1/10^10) * ![8193990033, 5298215096,
↪  1230614493]

noncomputable
def C1R : EuclideanSpace ℝ (Fin 3) := WithLp.toLp 2 (fun i =>
↪  C1 i)

noncomputable
def C2R : EuclideanSpace ℝ (Fin 3) := WithLp.toLp 2 (fun i =>
↪  C2 i)

noncomputable
def C3R : EuclideanSpace ℝ (Fin 3) := WithLp.toLp 2 (fun i =>
↪  C3 i)
```

Lemma 2.1.1

$\|C_1\| = 1$, $\frac{98}{100} < \|C_2\| < \frac{99}{100}$, and $\frac{98}{100} < \|C_3\| < \frac{99}{100}$.

Proof for Lemma 2.1.1

Trivial arithmetic.

Lean code for «code:c1_c2_c3_norms»

```
-- expose: {NopertInline.c1_norm_one,
↪  NopertInline.c2_norm_bound, NopertInline.c3_norm_bound,
↪  NopertInline.C15}

theorem c1_norm_one : ‖C1R‖ = 1 := by
  rw [EuclideanSpace.norm_eq]
  have lez : 0 ≤ ∑ i, ‖C1R i‖ ^ 2 := by
    apply Finset.sum_nonneg
    intro i _
    exact sq_nonneg (‖C1R i‖)
  rw [← Real.sq_sqrt lez]
  simp only [Real.norm_eq_abs, sq_abs]
  unfold C1R C1
  simp only [Fin.sum_univ_three, Pi.mul_apply,
   ↪  Matrix.cons_val]
  norm_num

theorem c2_norm_bound : ‖C2R‖ ∈ Set.Ioo (98/100) (99/100) :=
↪  by
  rw [EuclideanSpace.norm_eq]
  constructor
  · refine lt_sqrt_of_sq_lt ?_
    simp only [Real.norm_eq_abs, sq_abs]
    unfold C2R C2
    simp only [Fin.sum_univ_three, Pi.mul_apply,
     ↪  Matrix.cons_val]
    norm_num
  · refine (sqrt_lt' (by norm_num)).mpr ?_
    simp only [Real.norm_eq_abs, sq_abs]
    unfold C2R C2
    simp only [Fin.sum_univ_three, Pi.mul_apply,
     ↪  Matrix.cons_val]
    norm_num

theorem c2_norm_le_one : ‖C2R‖ ≤ 1 := by
  grw [c2_norm_bound.2]
  norm_num

theorem c3_norm_bound : ‖C3R‖ ∈ Set.Ioo (98/100) (99/100) :=
↪  by
  rw [EuclideanSpace.norm_eq]
  constructor
  · sorry
  · refine (sqrt_lt' (by norm_num)).mpr ?_
    simp only [Real.norm_eq_abs, sq_abs]
    unfold C3R C3
```

```
    simp only [Fin.sum_univ_three, Pi.mul_apply,
    ↪  Matrix.cons_val]
    norm_num

-- deps for c3_norm_bound:
--   { EuclideanSpace.norm_eq, lt_sqrt_of_sq_lt,
↪  Real.norm_eq_abs, C3R, C3,  }
-- deps we know: { }

theorem c3_norm_le_one : ‖C3R‖ ≤ 1 := by
  grw [c3_norm_bound.2]
  norm_num

/-- This is half of the C30 defined in \[SY25\]. In order
to see that this is pointsymmetric, it's convenient to
do explicit pointsymmetrization later. -/
noncomputable
def C15 (pt : ℝ³) : Finset ℝ³ :=
  Finset.range 15 |> .image fun (k : ℕ)  =>
    RzL (2 * π * (k : ℝ) / 15) pt

lemma C15_nonempty (pt : ℝ³) : (C15 pt).Nonempty := by
  use (RzL 0 pt)
  have z : 0 ∈ Finset.range 15 := Finset.insert_eq_self.mp rfl
  simp only [C15, Finset.mem_image, Finset.mem_range]
  use 0
  simp only [Nat.ofNat_pos, CharP.cast_eq_zero, mul_zero,
   ↪  zero_div, and_self]

lemma C15_pres_norm (pt v : ℝ³) (hv : v ∈ C15 pt) : ‖v‖ = ‖pt‖
↪  := by
  simp only [C15, Finset.mem_image, Finset.mem_range] at hv
  obtain ⟨a, ⟨ha, ha'⟩⟩ := hv
  rw [← ha', Bounding.Rz_preserves_norm _]

end NopertInline
```

Lemma 2.1.2

The radius of the Noperthedron is one.

Proof for Lemma 2.1.2

By Lemma 2.1.1, Theorem 4.3.3, «thm:polyhedron_radius_def», and Lemma 2.1.7.

Lean code for «code:lem:radius_noperthedron_one»

```
#check exactVertex_norm_le_one
#check exactPoly_point_symmetric
```

Rotations about the x, y, z axes $R_x, R_y, R_z : \mathbb{R} \to \mathbb{R}^{3\times 3}$ are defined in the usual way:

$$R_x(\alpha) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \qquad R_y(\alpha) := \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix},$$

$$R_z(\alpha) := \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Where Steininger and Yurkevich define a 30-element set C_{30}:

$$\mathcal{C}_{30} := \left\{ (-1)^{\ell} R_z\left(\frac{2\pi k}{15}\right) : k = 0, \ldots, 14; \ell = 0, 1 \right\}.$$

of rotations, we instead define

Definition 2.1.3

$$\mathcal{C}_{15} := \left\{ R_z\left(\frac{2\pi k}{15}\right) : k = 0, \ldots, 14 \right\}.$$

without point-symmetricness "baked in" as it is in C_{30}. It's more convenient for the formalization to apply C_{15} to the points C_1, C_2, C_3, and then point-symmetrize that set afterwards.

Definition 2.1.4

A set $S \subseteq \mathbb{R}^3$ is *point-symmetric* if $x \in S$ implies $-x \in S$.

Definition 2.1.5

The *pointsymmetrization* of a collection of vertices $v_1, \ldots, v_n \in \mathbb{R}^3$ is $v_1, \ldots, v_n, -v_1, \ldots, -v_n$.

We write $\mathcal{C}_{15} \cdot P = \{cP \text{ for } c \in \mathcal{C}_{15}\}$ for the orbit of P under the action of $\mathcal{C}_{15}$.

Definition 2.1.6

The Noperthedron is the polyhedron given by the vertex set that is the pointsymmetrization of

$$\mathcal{C}_{15} \cdot C_1 \cup \mathcal{C}_{15} \cdot C_2 \cup \mathcal{C}_{15} \cdot C_3$$

Lemma 2.1.7

The norm of any vertex in the prepointsymmetrized version of the Noperthedron is no more than 1.

Proof for Lemma 2.1.7

Evident from definitions.

Lemma 2.1.8

The pointsymmetrization of any set is point-symmetric.

Proof for Lemma 2.1.8

Evident from definitions.

Lemma 2.1.9

The noperthedron is point-symmetric.

Proof for Lemma 2.1.9

Follows from Lemma 2.1.8

2.2 Refined Rupert's property for the Noperthedron

Lemma 2.2.1

Let $\mathbf{P} = \mathbf{NOP}$, then for all $\theta, \varphi, \alpha \in \mathbb{R}$, the following three identities hold (as sets):

$$\begin{aligned} M(\theta + 2\pi/15, \varphi) \cdot \mathbf{P} &= M(\theta, \phi) \cdot \mathbf{P}, \\ R(\alpha + \pi) M(\theta, \phi) \cdot \mathbf{P} &= R(\alpha) M(\theta, \phi) \cdot \mathbf{P}, \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M(\theta, \phi) \cdot \mathbf{P} &= M(\theta + \pi/15, \pi - \varphi) \cdot \mathbf{P}. \end{aligned}$$

Proof for Lemma 2.2.1

See polyhedron.without.rupert, Lemma 7.

Corollary 2.2.2

If the noperthedron is Rupert, then there exists a solution with

$$\begin{aligned} \theta_1, \theta_2 &\in [0, 2\pi/15] \subset [0, 0.42], \\ \varphi_1 &\in [0, \pi] \subset [0, 3.15], \\ \varphi_2 &\in [0, \pi/2] \subset [0, 1.58], \\ \alpha &\in [-\pi/2, \pi/2] \subset [-1.58, 1.58]. \end{aligned}$$

Proof for Corollary 2.2.2

See polyhedron.without.rupert, Lemma 8.

3 Bounding Rotations

Lemma 3.1

For any $\alpha, \theta, \varphi \in \mathbb{R}$ and $a \in \{x, y, z\}$ one has $\|R(\alpha)\| = \|R_a(\alpha)\| = \|M(\theta, \phi)\| = 1$.

Proof for Lemma 3.1

See polyhedron.without.rupert, Lemma 9.

Lean code for «code:lem:RaRalpha»

```
theorem bp_Rx_norm_one (α : ℝ) : ‖RxL α‖ = 1 := by
  simpa using Bounding.Rx_norm_one α

theorem bp_Ry_norm_one (α : ℝ) : ‖RyL α‖ = 1 := by
  simpa using Bounding.Ry_norm_one α

theorem bp_Rz_norm_one (α : ℝ) : ‖RzL α‖ = 1 := by
  simpa using Bounding.Rz_norm_one α

theorem bp_rotR_norm_one (α : ℝ) : ‖rotR α‖ = 1 := by
  simpa using Bounding.rotR_norm_one α

theorem bp_rotM_norm_one (θ φ : ℝ) : ‖rotM θ φ‖ = 1 := by
  simpa using Bounding.rotM_norm_one θ φ
```

Lemma 3.2

Let $\epsilon > 0$, $|\alpha - \overline{\alpha}| \leq \varepsilon$ and $a \in \{x, y, z\}$ then $\|R_a(\alpha) - R_a(\overline{\alpha})\| = \|R(\alpha) - R(\overline{\alpha})\| < \varepsilon$.

Proof for Lemma 3.2

See polyhedron.without.rupert, Lemma 10.

Lean code for «code:lem:RaRa»

```
theorem bp_norm_rotR_sub_rotR_lt {ε α α_ : ℝ} (hε : 0 < ε) (hα
↪  : |α - α_| ≤ ε) :
    ‖rotR α - rotR α_‖ < ε := by
  simpa using Bounding.norm_rotR_sub_rotR_lt hε hα

theorem bp_norm_RxL_sub_RxL_eq {α α_ : ℝ} :
    ‖RxL α - RxL α_‖ = ‖rotR α - rotR α_‖ := by
  simpa using Bounding.norm_RxL_sub_RxL_eq (α := α) (α_ := α_)

theorem bp_norm_RyL_sub_RyL_eq {α α_ : ℝ} :
    ‖RyL α - RyL α_‖ = ‖rotR α - rotR α_‖ := by
  simpa using Bounding.norm_RyL_sub_RyL_eq (α := α) (α_ := α_)

theorem bp_norm_RzL_sub_RzL_eq {α α_ : ℝ} :
    ‖RzL α - RzL α_‖ = ‖rotR α - rotR α_‖ := by
  simpa using Bounding.norm_RzL_sub_RzL_eq (α := α) (α_ := α_)
```

Lemma 3.3

For any $\alpha, \beta \in \mathbb{R}$ one has $\|R_x(\alpha)R_y(\beta) - \mathrm{id}\| \leq \sqrt{\alpha^2 + \beta^2}$ with equality only for $\alpha = \beta = 0$.

Proof for Lemma 3.3

The composition $R_d(\alpha)R_{d'}(\beta)$ is a rotation, i.e. conjugate to some $R_z(\gamma)$ by an isometry, which gives $\|R_d(\alpha)R_{d'}(\beta) - \mathrm{id}\|^2 = 2(1 - \cos\gamma) = 3 - \mathrm{tr}(R_d(\alpha)R_{d'}(\beta)) = 3 - (\cos\alpha + \cos\beta + \cos\alpha\cos\beta)$. Writing the right hand side as $2(1-\cos\alpha) + 2(1-\cos\beta) - (1-\cos\alpha)(1-\cos\beta)$ and using $2(1-\cos x) \leq x^2$ (with equality only at $x = 0$) yields the bound and the equality condition. This is a direct route to polyhedron.without.rupert, Lemma 12, that avoids the Jensen-inequality argument of Lemma 11 and the doubling induction.

Lean code for «code:lem:RxRy»

```
theorem bp_lemma12 {d d' : Fin 3} {α β : ℝ} (hd : d ≠ d') :
    ‖rot3 d α ∘L rot3 d' β - 1‖ ≤ √(α ^ 2 + β ^ 2) := by
  simpa using Bounding.lemma12 (d := d) (d' := d') (α := α) (β
  ↪  := β) hd

theorem bp_lemma12_equality_iff {d d' : Fin 3} {α β : ℝ} (hd :
↪  d ≠ d') :
    ‖rot3 d α ∘L rot3 d' β - 1‖ = √(α ^ 2 + β ^ 2) ↔ (α = 0 ∧
    ↪  β = 0) := by
  simpa using Bounding.lemma12_equality_iff (d := d) (d' :=
  ↪  d') (α := α) (β := β) hd
```

Lemma 3.4

Let $\epsilon > 0$ and $|\theta-\overline{\theta}|, |\varphi-\overline{\varphi}| \le \varepsilon$ then $\|M(\theta,\phi)-M(\overline{\theta},\overline{\varphi})\|, \|X(\theta,\varphi)-X(\overline{\theta},\overline{\varphi})\| < \sqrt{2}\varepsilon$.

Proof for Lemma 3.4

See polyhedron.without.rupert, Lemma 13.

Lean code for «code:lem:sqrt2»

```
theorem bp_RyL_neg_compose_RyL (α : ℝ) : RyL (-α) ∘L RyL α =
↪  ContinuousLinearMap.id _ _ := by
  simpa using Bounding.RyL_neg_compose_RyL (α := α)

theorem bp_RzL_neg_compose_RzL (α : ℝ) : RzL (-α) ∘L RzL α =
↪  ContinuousLinearMap.id _ _ := by
  simpa using Bounding.RzL_neg_compose_RzL (α := α)

theorem bp_norm_M_sub_lt {ε θ θ_ φ φ_ : ℝ} (hε : 0 < ε) (hθ :
↪   |θ - θ_| ≤ ε) (hφ : |φ - φ_| ≤ ε) :
    ‖rotM θ φ - rotM θ_ φ_‖ < √2 * ε := by
  simpa using Bounding.norm_M_sub_lt hε hθ hφ

theorem bp_norm_X_sub_lt {ε θ θ_ φ φ_ : ℝ} (hε : 0 < ε) (hθ :
↪   |θ - θ_| ≤ ε) (hφ : |φ - φ_| ≤ ε) :
    ‖vecX θ φ - vecX θ_ φ_‖ < √2 * ε := by
  simpa using Bounding.norm_X_sub_lt hε hθ hφ
```

Lemma 3.5

Let $P \in \mathbb{R}^3$ with $\|P\| \le 1$. Further, let $\epsilon > 0$ and $\overline{\theta}, \overline{\varphi}, \theta, \phi \in \mathbb{R}$ such that $|\overline{\theta}-\theta|, |\overline{\varphi}-\phi| \le \epsilon$. If $\langle X(\overline{\theta},\overline{\varphi}), P\rangle > \sqrt{2}\varepsilon$ then $\langle X(\theta,\phi), P\rangle > 0$.

Proof for Lemma 3.5

See polyhedron.without.rupert, Lemma 14.

Lean code for «code:lem:XPgt0»

```
theorem bp_XPgt0 {P : ℝ³} {ε θ θ_ φ φ_ : ℝ} (hP : ‖P‖ ≤ 1) (hε
↪   : 0 < ε)
    (hθ : |θ - θ_| ≤ ε) (hφ : |φ - φ_| ≤ ε)
    (hX : √2 * ε < ⟪vecX θ_ φ_, P⟫) :
    0 < ⟪vecX θ φ, P⟫ := by
  simpa using Bounding.XPgt0 hP hε hθ hφ hX
```

Lemma 3.6

Let $P \in \mathbb{R}^3$ with $\|P\| \le 1$. Further, let $\epsilon, r > 0$ and $\overline{\theta}, \overline{\varphi}, \theta, \phi \in \mathbb{R}$ such that $|\overline{\theta}-\theta|, |\overline{\varphi}-\phi| \le \epsilon$. If $\|M(\overline{\theta},\overline{\varphi})P\| > r + \sqrt{2}\varepsilon$ then $\|M(\theta,\phi)P\| > r$.

Proof for Lemma 3.6

See polyhedron.without.rupert, Lemma 15. Corrigendum: the triangle inquality only implies greater than **or equal to**.

Lean code for «code:lem:MPgtr»

```
theorem bp_norm_M_apply_gt {ε r θ θ_ φ φ_ : ℝ} {P : ℝ³} (hP :
↪  ‖P‖ ≤ 1) (hε : 0 < ε)
    (hθ : |θ - θ_| ≤ ε) (hφ : |φ - φ_| ≤ ε)
    (hM : r + √2 * ε < ‖rotM θ_ φ_ P‖) :
    r < ‖rotM θ φ P‖ := by
  simpa using Bounding.norm_M_apply_gt hP hε hθ hφ hM
```

Lemma 3.7

Let $\epsilon > 0$ and $|\theta - \overline{\theta}|, |\varphi - \overline{\varphi}|, |\alpha - \overline{\alpha}| \leq \varepsilon$ then $\|R(\alpha)M(\theta, \phi) - R(\overline{\alpha})M(\overline{\theta}, \overline{\varphi})\| < \sqrt{5}\varepsilon$.

Proof for Lemma 3.7

See polyhedron.without.rupert, Lemma 16.

Lean code for «code:lem:sqrt5»

```
theorem bp_RyL_neg_compose_RyL' (α : ℝ) : RyL (-α) ∘L RyL α =
↪  ContinuousLinearMap.id _ _ := by
  simpa using Bounding.RyL_neg_compose_RyL (α := α)

theorem bp_RzL_neg_compose_RzL' (α : ℝ) : RzL (-α) ∘L RzL α =
↪  ContinuousLinearMap.id _ _ := by
  simpa using Bounding.RzL_neg_compose_RzL (α := α)

theorem bp_norm_RM_sub_RM_le {ε θ θ_ φ φ_ α α_ : ℝ} (hε : 0 <
↪  ε) (hθ : |θ - θ_| ≤ ε) (hφ : |φ - φ_| ≤ ε)
    (hα : |α - α_| ≤ ε) :
    ‖rotprojRM θ φ α - rotprojRM θ_ φ_ α_‖ < √5 * ε := by
  simpa using Bounding.norm_RM_sub_RM_le hε hθ hφ hα
```

4 Preliminaries

TODO: This whole chapter needs organization, it's just a grab bag of miscellaneous results for now.

4.1 Rupert Sets

Theorem 4.1.1

The following are equivalent:

- The convex polyhedron with vertex set v is Rupert.
- The convex closure of v is a Rupert set.

Proof for Theorem 4.1.1

TODO: import this from the other repo

4.2 Poses

TODO
Theorem 4.2.1

Given a pose with zero offset, there exists a 5-parameter pose that is equivalent to it.

Proof for Theorem 4.2.1

By putting the pose into a canonical form as a Z rotation followed by a Y followed by a Z.

4.3 Pointsymmetry and Rupertness

Theorem 4.3.1

If a set is point symmetric and convex, then it being Rupert implies it being purely rotationally Rupert.

Proof for Theorem 4.3.1

TODO: informalize proof

Theorem 4.3.2

Suppose S is a finite set of points in $\mathbb{R}^n$. The radius of the polyhedron S is r iff

- there is a vector $v \in S$ with $\|v\| = r$
- all vectors $v \in S$ have $\|v\| \leq r$

Proof for Theorem 4.3.2

Immediate from definition.

Theorem 4.3.3

Pointsymmetrization preserves radius.

Proof for Theorem 4.3.3

Because the reflection of a point about the origin preserves its norm.

5 The Global Theorem

Lemma 5.1

Suppose $V = V_1, \dots, V_m \subseteq \mathbb{R}^n$ is a finite sequence of points, and let $\mathsf{Co}(V)$ be its convex hull. If $S \in \mathsf{Co}(V)$ and $w \in \mathbb{R}^n$, then

$$\langle S, w\rangle \leq \max_i \langle V_i, w\rangle.$$

Proof for Lemma 5.1

This is a mild generalization of Steininger and Yurkevich (2025), Lemma 18.Since $S \in \mathsf{Co}(V)$, we have $S = \sum_{j=1}^m \lambda_j V_j$ for some $\lambda_1, \dots, \lambda_m \in [0,1]$ with $1 = \sum_{j=1}^m \lambda_j$. Therefore

$$\langle S, w\rangle = \left\langle \sum_{j=1}^m \lambda_j V_j, w \right\rangle = \sum_{j=1}^m \lambda_j \langle V_j, w\rangle \leq \sum_{j=1}^m \lambda_j \max_i \langle V_i, w\rangle$$

$$= \max_i \langle V_i, w\rangle \sum_{j=1}^m \lambda_j = \max_i \langle V_i, w\rangle$$

as required.

Lemma 5.2

Let $S \in \mathbb{R}^3$ and $w \in \mathbb{R}^2$ be unit vectors, and set $f(x_1, x_2, x_3) = \langle R(x_3) M(x_1, x_2) S, w\rangle$. Then for all $x_1, x_2, x_3 \in \mathbb{R}$ and any $i, j \in \{1, 2, 3\}$ it holds that

$$\left| \frac{\partial^2 f}{\partial x_i \partial x_j}(x_1, x_2, x_3) \right| \leq 1.$$

Proof for Lemma 5.2

See polyhedron.without.rupert, Lemma 19.

Lemma 5.3

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2-function and $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}$ such that $|x_1 - y_1|, \dots, |x_n - y_n| \le \varepsilon$. If $|\partial_{x_i}\partial_{x_j} f(v)| \le 1$ for all $i, j \in \{1, \dots, n\}$ and all $v \in \mathbb{R}^n$, then

$$|f(x_1, \dots, x_n) - f(y_1, \dots, y_n)| \le \varepsilon \sum_{i=1}^{n} |\partial_{x_i} f(x_1, \dots, x_n)| + \frac{n^2}{2}\varepsilon^2.$$

Proof for Lemma 5.3

See polyhedron.without.rupert, Lemma 20.

Lemma 5.4

The partial derivatives of all relevant rotations, projections, and inner products used in the Global Theorem are as expected. Specifically:

- $f^\alpha(\theta, \phi, \alpha) = \langle R'(\alpha) M(\theta, \phi) S, w \rangle$
- $f^\theta(\theta, \phi, \alpha) = \langle R(\alpha) M^\theta(\theta, \phi) S, w \rangle$
- $f^\phi(\theta, \phi, \alpha) = \langle R(\alpha) M^\phi(\theta, \phi) S, w \rangle$
- $g^\theta(\theta, \phi) = \langle M^\theta(\theta, \phi) P, w \rangle$
- $g^\phi(\theta, \phi) = \langle M^\phi(\theta, \phi) P, w \rangle$

where $f(\theta, \phi, \alpha) = \langle R(\alpha) M(\theta, \phi) S/\|S\|, w \rangle$ and $g(\theta, \phi) = \langle M(\theta, \phi) P/\|P\|, w \rangle$.

Proof for Lemma 5.4

By basic properties of derivatives.

Theorem 5.5

Let $\mathbf{P}$ be a pointsymmetric convex polyhedron with radius $\rho = 1$ and let $S \in \mathbf{P}$. Further let $\overline{\theta}_1, \overline{\varphi}_1, \overline{\theta}_2, \overline{\varphi}_2, \overline{\alpha} \in \mathbb{R}$ and let $w \in \mathbb{R}^2$ be a unit vector. Denote $\overline{M_1} := M(\overline{\theta}_1, \overline{\varphi}_1)$, $\overline{M_2} := M(\overline{\theta}_2, \overline{\varphi}_2)$ as well as $\overline{M_1}^\theta := M^\theta(\overline{\theta}_1, \overline{\varphi}_1)$, $\overline{M_1}^\phi := M^\phi(\overline{\theta}_1, \overline{\varphi}_1)$ and analogously for $\overline{M_2}^\theta, \overline{M_2}^\phi$. Finally set

$$G := \langle R(\overline{\alpha})\overline{M_1}S, w\rangle - \epsilon \cdot \left(|\langle R'(\overline{\alpha})\overline{M_1}S, w\rangle| + |\langle R(\overline{\alpha})\overline{M_1}^\theta S, w\rangle| + |\langle R(\overline{\alpha})\overline{M_1}^\phi S, w\rangle|\right) - 9\epsilon^2/2,$$

$$H_P := \langle \overline{M_2}P, w\rangle + \epsilon \cdot \left(|\langle \overline{M_2}^\theta P, w\rangle| + |\langle \overline{M_2}^\varphi P, w\rangle|\right) + 2\epsilon^2, \quad \text{for } P \in \mathbf{P}.$$

If $G > \max_{P \in \mathbf{P}} H_P$ then there does not exist a solution to Rupert's condition with

$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in U := [\overline{\theta}_1 \pm \epsilon, \overline{\varphi}_1 \pm \epsilon, \overline{\theta}_2 \pm \epsilon, \overline{\varphi}_2 \pm \epsilon, \overline{\alpha} \pm \epsilon] \subseteq \mathbb{R}^5.$$

Proof for Theorem 5.5

See polyhedron.without.rupert, Section 4.2.

6 The Local Theorem

Lemma 6.1

For any $P \in \mathbb{R}^3$ one has $\|M(\theta,\phi)P\|^2 = \|P\|^2 - \langle X(\theta,\varphi), P\rangle^2$.

Proof for Lemma 6.1

See polyhedron.without.rupert, Lemma 21.

Definition 6.2

Given $v_1, \dots, v_n \in \mathbb{R}^n$ write $\text{span}^+(v_1, \dots, v_n)$ for the set (simplicial cone) in $\mathbb{R}^n$ defined by

$$\text{span}^+(v_1, \dots, v_n) = \Big\{ w \in \mathbb{R}^n \colon \exists \lambda_1, \dots, \lambda_n > 0 \text{ s.t. } w = \sum_{i=1}^{n} \lambda_i v_i \Big\},$$

which is the natural restriction of $\text{span}(v_1, \dots, v_n)$ to positive weights.

Lemma 6.3

Let $V_1, V_2, V_3, Y, Z \in \mathbb{R}^3$ with $\|Y\| = \|Z\|$ and $Y, Z \in \text{span}^+(V_1, V_2, V_3)$. Then at least one of the following inequalities fails:

- $\langle V_1, Y\rangle > \langle V_1, Z\rangle$
- $\langle V_2, Y\rangle > \langle V_2, Z\rangle$
- $\langle V_3, Y\rangle > \langle V_3, Z\rangle$

Proof for Lemma 6.3

See polyhedron.without.rupert, Lemma 23.

Lemma 6.4

For $A, \overline{A}, B, \overline{B} \in \mathbb{R}^{n\times n}$ and $P_1, P_2 \in \mathbb{R}^n$ it holds that

$$|\langle AP_1, BP_2\rangle - \langle \overline{A}P_1, \overline{B}P_2\rangle| \leq \|P_1\|\,\|P_2\|\,\big(\|A-\overline{A}\|\,\|\overline{B}\| + \|\overline{A}\|\,\|B-\overline{B}\| + \|A-\overline{A}\|\,\|B-\overline{B}\|\big).$$

Proof for Lemma 6.4

See polyhedron.without.rupert, Lemma 24.

Lemma 6.5

For $A, B \in \mathbb{R}^{n\times n}$ and $P_1, P_2 \in \mathbb{R}^n$ one has

$|\langle AP_1, AP_2\rangle - \langle BP_1, BP_2\rangle| \leq \|P_1\|\,\|P_2\|\,\|A-B\|\,(\|A\|+\|B\|+\|A-B\|)$.

Proof for Lemma 6.5

See polyhedron.without.rupert, Lemma 25.

Lemma 6.6

Let $A, B, C \in \mathbb{R}^2$ be such that $\langle R(\pi/2)A, B\rangle$, $\langle R(\pi/2)B, C\rangle$, $\langle R(\pi/2)C, A\rangle > 0$. Then the origin lies strictly in triangle ABC.

Proof for Lemma 6.6

See polyhedron.without.rupert, Lemma 26.

Definition 6.7

Let $\theta, \varphi \in \mathbb{R}$, $\varepsilon > 0$, and set $M := M(\theta, \varphi)$. Three points $P_1, P_2, P_3 \in \mathbb{R}^3$ with $\|P_1\|, \|P_2\|, \|P_3\| \leq 1$ are called ε-spanning for (θ, φ) if:

- $\langle R(\pi/2)MP_1, MP_2\rangle > 2\epsilon(\sqrt{2}+\varepsilon)$
- $\langle R(\pi/2)MP_2, MP_3\rangle > 2\epsilon(\sqrt{2}+\varepsilon)$
- $\langle R(\pi/2)MP_3, MP_1\rangle > 2\epsilon(\sqrt{2}+\varepsilon)$

Lemma 6.8

Let $P_1, P_2, P_3 \in \mathbb{R}^3$ with $\|P_1\|, \|P_2\|, \|P_3\| \leq 1$ be ϵ-spanning for $(\bar\theta, \bar\phi)$ and let $\theta, \phi \in \mathbb{R}$ satisfy $|\theta - \bar\theta|, |\phi - \bar\phi| \leq \epsilon$. If $\langle X(\theta, \phi), P_i\rangle > 0$ for $i = 1, 2, 3$, then $X(\theta, \phi) \in \text{span}^+(P_1, P_2, P_3)$.

Proof for Lemma 6.8

See polyhedron.without.rupert, Lemma 28.

Lemma 6.9

Let $P, Q \in \mathbb{R}^3$ with $\|P\|, \|Q\| \leq 1$, and let $\epsilon > 0$, $\bar\theta_1, \bar\phi_1, \bar\theta_2, \bar\phi_2, \bar\alpha \in \mathbb{R}$. Set

$$T := \left(R(\bar\alpha)M(\bar\theta_1, \bar\phi_1)P + M(\bar\theta_2, \bar\phi_2)Q\right)/2 \in \mathbb{R}^2,$$

and assume $\delta \geq \|T - M(\bar\theta_2, \bar\phi_2)Q\|$. If $|\bar\theta_1 - \theta_1|, |\bar\phi_1 - \phi_1|, |\bar\theta_2 - \theta_2|, |\bar\phi_2 - \phi_2|, |\bar\alpha - \alpha| \leq \epsilon$, then $R(\alpha)M(\theta_1, \phi_1)P$ and $M(\theta_2, \phi_2)Q$ lie in $\text{Circ}_{\delta+\sqrt{5}\epsilon}(T)$.

Proof for Lemma 6.9

See polyhedron.without.rupert, Lemma 30.

Definition 6.10

Let $\mathcal{P} \subset \mathbb{R}^2$ be a convex polygon and $Q \in \mathcal{P}$ one of its vertices. Assume that for some $\overline{Q} \in \mathbb{R}^2$ it holds that $Q \in \mathrm{Disc}_\delta(\overline{Q})$, i.e. $\|Q - \overline{Q}\| < \delta$. Define $\mathrm{Sect}_\delta(\overline{Q}) := \mathrm{Disc}_\delta(\overline{Q}) \cap \mathcal{P}^\circ$ as the intersection between $\mathrm{Disc}_\delta(\overline{Q})$ and the interior of the convex hull of $\mathcal{P}$.Moreover, $Q \in \mathcal{P}$ is called δ-locally maximally distant with respect to $\overline{Q}$ (δ-LMD($\overline{Q}$)) if for all $A \in \mathrm{Sect}_\delta(\overline{Q})$ it holds that $\|Q\| > \|A\|$.

Lemma 6.11

Let **P** be a convex polygon and $Q \in \mathbf{P}$ a vertex. Let $\overline{Q} \in \mathbb{R}^2$ with $\|Q - \overline{Q}\| < \delta$ for some $\delta > 0$. Assume there exists $r > 0$ with $\|Q\| > r$ such that

$$\frac{\langle Q, Q - P_j\rangle}{\|Q\|\|Q - P_j\|} \geq \delta/r$$

for all other vertices $P_j \in \mathbf{P} \setminus Q$. Then Q is δ-locally maximally distant with respect to $\overline{Q}$.

Proof for Lemma 6.11

See polyhedron.without.rupert, Lemma 32.

Lemma 6.12

Let $\epsilon > 0$ and $\theta, \bar{\theta}, \phi, \bar{\phi} \in \mathbb{R}$ with $|\theta - \bar{\theta}|, |\phi - \bar{\phi}| \leq \epsilon$. Define $M = M(\theta, \phi)$ and $\overline{M} = M(\bar{\theta}, \bar{\phi})$, and let $P, Q \in \mathbb{R}^3$ with $\|P\|, \|Q\| \leq 1$. Assume that

$$\frac{\langle \overline{M}P, \overline{M}(P - Q)\rangle - 2\epsilon\|P - Q\| \cdot (\sqrt{2} + \varepsilon)}{\left(\|\overline{M}P\| + \sqrt{2}\varepsilon\right) \cdot \left(\|\overline{M}(P - Q)\| + 2\sqrt{2}\varepsilon\right)} > 0.$$

Then:

$$\frac{\langle MP, M(P - Q)\rangle}{\|MP\|\,\|M(P - Q)\|} \geq \frac{\langle \overline{M}P, \overline{M}(P - Q)\rangle - 2\epsilon\|P - Q\| \cdot (\sqrt{2} + \varepsilon)}{(\|\overline{M}P\| + \sqrt{2}\varepsilon) \cdot (\|\overline{M}(P - Q)\| + 2\sqrt{2}\varepsilon)}.$$

Proof for Lemma 6.12

See polyhedron.without.rupert, Lemma 33.

Lemma 6.13

Let $P_1, P_2, P_3, Q_1, Q_2, Q_3 \in \mathbb{R}^3$. Define $P := (P_1|P_2|P_3)$ and $Q := (Q_1|Q_2|Q_3)$ and assume Q is invertible. Then P_1, P_2, P_3 and Q_1, Q_2, Q_3 are congruent iff $P^tP = Q^tQ$.

Proof for Lemma 6.13

From polyhedron.without.rupert, Lemma 35. Note that $P^tP = Q^tQ$ is equivalent to saying that $\langle P_i, P_j\rangle = \langle Q_i, Q_j\rangle$ for all $1 \leq i, j \leq 3$. Moreover, the condition on invertibility of Q can be dropped, however then the proof becomes somewhat less straightforward.If P_1, P_2, P_3 and Q_1, Q_2, Q_3 are congruent then $\langle P_i, P_j\rangle = \langle LQ_i, LQ_j\rangle = \langle Q_i, Q_j\rangle$, thus $P^tP = Q^tQ$. For the other direction, we claim that $L := PQ^{-1}$ is orthonormal and satisfies that $P_i = LQ_i$ for all $i = 1, 2, 3$. Indeed, $L^tL = (PQ^{-1})^t(PQ^{-1}) = (Q^t)^{-1}P^tPQ^{-1} = \mathrm{Id}$ and it holds that $LQ = PQ^{-1}Q = P$, thus $LQ_i = P_i$.

Theorem 6.14

Let **P** be a polyhedron with radius $\rho = 1$ and $P_1, P_2, P_3, Q_1, Q_2, Q_3 \in$ **P** be not necessarily distinct. Assume that P_1, P_2, P_3 and Q_1, Q_2, Q_3 are congruent.Let $\epsilon > 0$ and $\overline{\theta}_1, \overline{\varphi}_1, \overline{\theta}_2, \overline{\varphi}_2, \overline{\alpha} \in \mathbb{R}$, then set $\overline{X_1} := X(\overline{\theta}_1, \overline{\varphi}_1), \overline{X_2} := X(\overline{\theta}_2, \overline{\varphi}_2)$ as well as $\overline{M_1} := M(\overline{\theta}_1, \overline{\varphi}_1), \overline{M_2} := M(\overline{\theta}_2, \overline{\varphi}_2)$. Assume that there exist $\sigma_P, \sigma_Q \in \{0, 1\}$ such that

$$(-1)^{\sigma_P}\langle\overline{X_1}, P_i\rangle > \sqrt{2}\varepsilon \quad \text{and} \quad (-1)^{\sigma_Q}\langle\overline{X_2}, Q_i\rangle > \sqrt{2}\varepsilon,$$

for all $i = 1, 2, 3$. Moreover, assume that P_1, P_2, P_3 are ϵ-spanning for $(\overline{\theta}_1, \overline{\varphi}_1)$ and that Q_1, Q_2, Q_3 are ϵ-spanning for $(\overline{\theta}_2, \overline{\varphi}_2)$. Finally, assume that for all $i = 1, 2, 3$ and any $Q_j \in \mathbf{P} \setminus Q_i$ it holds that

$$\frac{\langle\overline{M_2}Q_i, \overline{M_2}(Q_i - Q_j)\rangle - 2\epsilon\|Q_i - Q_j\| \cdot (\sqrt{2} + \varepsilon)}{\left(\|\overline{M_2}Q_i\| + \sqrt{2}\varepsilon\right) \cdot \left(\|\overline{M_2}(Q_i - Q_j)\| + 2\sqrt{2}\varepsilon\right)} > \frac{\sqrt{5}\epsilon + \delta}{r},$$

for some $r > 0$ such that $\min_{i=1,2,3} \|\overline{M_2}Q_i\| > r + \sqrt{2}\epsilon$ and for some $\delta \in \mathbb{R}$ with

$$\delta \geq \max_{i=1,2,3} \left\|R(\overline{\alpha})\overline{M_1}P_i - \overline{M_2}Q_i\right\|/2.$$

Then there exists no solution to Rupert's problem $R(\alpha)M(\theta_1, \phi_1)\mathbf{P} \subset M(\theta_2, \phi_2)\mathbf{P}^\circ$ with

$$(\theta_1, \phi_1, \theta_2, \phi_2, \alpha) \in [\overline{\theta}_1 \pm \epsilon, \overline{\varphi}_1 \pm \epsilon, \overline{\theta}_2 \pm \epsilon, \overline{\varphi}_2 \pm \epsilon, \overline{\alpha} \pm \epsilon] := U \subseteq \mathbb{R}^5.$$

Proof for Theorem 6.14

See polyhedron.without.rupert, Theorem 36.

7 Rational Versions

Definition 7.1

Define $\sin_{\mathbb{Q}}, \cos_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ by truncated Taylor series:

- $\sin_{\mathbb{Q}}(x) := x - \frac{x^3}{3!} + \frac{x^5}{5!} \mp \cdots + \frac{x^{25}}{25!}$
- $\cos_{\mathbb{Q}}(x) := 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \mp \cdots + \frac{x^{24}}{24!}$

Replacing sin, cos with $\sin_{\mathbb{Q}}, \cos_{\mathbb{Q}}$ defines $R_{\mathbb{Q}}(\alpha), R'_{\mathbb{Q}}(\alpha), X_{\mathbb{Q}}(\theta, \phi), M_{\mathbb{Q}}(\theta, \phi), M^{\theta}_{\mathbb{Q}}(\theta, \varphi), M^{\phi}_{\mathbb{Q}}(\theta, \varphi)$.

Lemma 7.2

$$|\sin_{\mathbb{Q}}(x) - \sin(x)| \leq \frac{|x|^{27}}{27!} \qquad \text{and} \qquad |\cos_{\mathbb{Q}}(x) - \cos(x)| \leq \frac{|x|^{26}}{26!}.$$

Proof for Lemma 7.2

Appeal to Taylor series bounds, using the fact that all absolute values of higher derivatives of sine and cosine never exceed 1.

Lemma 7.3

For every $x \in [-4, 4]$, $|\sin_{\mathbb{Q}}(x) - \sin(x)| \leq \kappa/7$ and $|\cos_{\mathbb{Q}}(x) - \cos(x)| \leq \kappa/7$.

Proof for Lemma 7.3

Straightforward numerical calculation from Lemma Lemma 7.2.

Lemma 7.4

Let $A = (a_{i,j})_{1 \leq i \leq m,\, 1 \leq j \leq n} \in \mathbb{R}^{m \times n}$ and $\delta > 0$. If $|a_{i,j}| \leq \delta$ for all entries, then $\|A\| \leq \delta\sqrt{mn}$.

Proof for Lemma 7.4

For any $v \in \mathbb{R}^n$ we have

$$\|Av\|^2 = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{i,j} v_j \right)^2 \leq \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \delta |v_j| \right)^2 = \delta^2 m \left(\sum_{j=1}^{n} |v_j| \right)^2 \leq \delta^2 mn \|v\|^2$$

using the Cauchy-Schwarz inequality. Dividing by $\|v\|$ and taking the square root proves the claim.

Lemma 7.5

Let $A(x, y)$ be an $m \times n$ matrix $(1 \leq m, n \leq 3)$ whose entries are products of terms in $\{0, 1, -1, \pm\sin(z), \pm\cos(z)\}$. Let $A_{\mathbb{Q}}(x, y)$ be obtained by replacing sin, cos with $\sin_{\mathbb{Q}}, \cos_{\mathbb{Q}}$. Then for $x, y \in [-4, 4]$, $\|A(x, y) - A_{\mathbb{Q}}(x, y)\| \leq \kappa$.

Proof for Lemma 7.5

We've replaced the assumption $a_i(z) \in \{0, 1, -1, \pm\sin(z), \pm\cos(z)\}$ in Steininger and Yurkevich (2025)'s Lemma 40 with $a_i(z) \in [-1, 1]$.By assumption, for fixed x, y every entry of $A(x, y) - A_{\mathbb{Q}}(x, y)$ is of the form $ab - \widetilde{a}\widetilde{b}$ for some $a, b \in [-1, 1]$ and $|a - \widetilde{a}|, |b - \widetilde{b}| \leq \kappa/7$ by Lemma 7.3. This implies that

$$\begin{aligned} |ab - \widetilde{a}\widetilde{b}| &\leq |ab - a\widetilde{b}| + |a\widetilde{b} - \widetilde{a}\widetilde{b}| \\ &= |a| \cdot |b - \widetilde{b}| + |\widetilde{b}| \cdot |a - \widetilde{a}| \leq 1 \cdot \kappa/7 + (1 + \kappa/7) \cdot \kappa/7 < \kappa/3. \end{aligned}$$

So we can apply Lemma 7.4 and obtain that $\|A(x, y) - A_{\mathbb{Q}}(x, y)\| < \kappa/3 \cdot \sqrt{3 \cdot 3} = \kappa$.

Corollary 7.6

Let $\alpha, \theta, \phi \in [-4, 4]$. Then $\|R(\alpha) - R_{\mathbb{Q}}(\alpha)\|$, $\|R'(\alpha) - R'_{\mathbb{Q}}(\alpha)\|$, $\|X(\theta, \phi) - X_{\mathbb{Q}}(\theta, \phi)\|$, $\|M(\theta, \phi) - M_{\mathbb{Q}}(\theta, \phi)\|$, $\|M^\theta(\theta, \phi) - M^\theta_{\mathbb{Q}}(\theta, \phi)\|$, $\|M^\phi(\theta, \phi) - M^\phi_{\mathbb{Q}}(\theta, \phi)\|$ are all at most κ. Moreover $\|R_{\mathbb{Q}}(\alpha)\|, \|R'_{\mathbb{Q}}(\alpha)\|, \|M_{\mathbb{Q}}(\theta, \phi)\|, \|M^\theta_{\mathbb{Q}}(\theta, \varphi)\|, \|M^\phi_{\mathbb{Q}}$ $1 + \kappa$.

Proof for Corollary 7.6

The first statement is a direct application of Lemma 7.5 and the second statement follows immediately after using Lemma 3.1 and the triangle inequality. The derivative norm bounds follow similarly, using that the operator norms of R', M^θ, and M^ϕ are at most 1.

Lemma 7.7

Let (A_i, B_i) for $1 \leq i \leq n$ be pairs of same-size real matrices, with products $A_1 \cdots A_n$ and $B_1 \cdots B_n$ defined. If $\|A_i - B_i\| \leq \kappa$ and $\delta_i \geq \max(\|A_i\|, \|B_i\|, 1)$, then $\|A_1 \cdots A_n - B_1 \cdots B_n\| \leq n\kappa\, \delta_1 \cdots \delta_n$.

Proof for Lemma 7.7

See polyhedron.without.rupert, Lemma 42.

Lemma 7.8

Let $\alpha, \theta, \phi \in [-4, 4]$, $P \in \mathbb{R}^3$ with $\|P\| \leq 1$ and let $\widetilde{P}$ be a κ-rational approximation of P. Set $M = M(\theta, \phi)$ and $M_{\mathbb{Q}} = M_{\mathbb{Q}}(\theta, \phi)$, $M^\theta = M^\theta(\theta, \phi)$ and $M^\theta_{\mathbb{Q}} = M^\theta_{\mathbb{Q}}(\theta, \phi)$, $M^\phi = M^\phi(\theta, \phi)$ and $M^\phi_{\mathbb{Q}} = M^\phi_{\mathbb{Q}}(\theta, \phi)$, as well as $R = R(\alpha)$, $R_{\mathbb{Q}} = R_{\mathbb{Q}}(\alpha)$, $R' = R'(\alpha)$, $R'_{\mathbb{Q}} = R'_{\mathbb{Q}}(\alpha)$. Finally let $w \in \mathbb{R}^2$ with $\|w\| = 1$. Then:

$$|\langle MP, w\rangle - \langle M_{\mathbb{Q}}\widetilde{P}, w\rangle| \leq 3\kappa \tag{7.1}$$
$$|\langle M^\theta P, w\rangle - \langle M^\theta_{\mathbb{Q}}\widetilde{P}, w\rangle| \leq 3\kappa, \tag{7.2}$$
$$|\langle M^\phi P, w\rangle - \langle M^\phi_{\mathbb{Q}}\widetilde{P}, w\rangle| \leq 3\kappa, \tag{7.3}$$
$$|\langle RMP, w\rangle - \langle R_{\mathbb{Q}}M_{\mathbb{Q}}\widetilde{P}, w\rangle| \leq 4\kappa \tag{7.4}$$
$$|\langle R'MP, w\rangle - \langle R'_{\mathbb{Q}}M_{\mathbb{Q}}\widetilde{P}, w\rangle| \leq 4\kappa, \tag{7.5}$$
$$|\langle RM^\theta P, w\rangle - \langle R_{\mathbb{Q}}M^\theta_{\mathbb{Q}}\widetilde{P}, w\rangle| \leq 4\kappa, \tag{7.6}$$
$$|\langle RM^\phi P, w\rangle - \langle R_{\mathbb{Q}}M^\phi_{\mathbb{Q}}\widetilde{P}, w\rangle| \leq 4\kappa. \tag{7.7}$$

Proof for Lemma 7.8

See polyhedron.without.rupert, Lemma 44.

Theorem 7.9

Let $\mathbf{P}$ be a pointsymmetric convex polyhedron with radius $\rho = 1$ and $\widetilde{\mathbf{P}}$ a κ-rational approximation. Let $\widetilde{S} \in \widetilde{\mathbf{P}}$. Further let $\epsilon > 0$ and $\overline{\theta}_1, \overline{\varphi}_1, \overline{\theta}_2, \overline{\varphi}_2, \overline{\alpha} \in \mathbb{Q} \cap [-4, 4]$. Let $w \in \mathbb{Q}^2$ be a unit vector. Denote $\overline{M_1} := M_{\mathbb{Q}}(\overline{\theta}_1, \overline{\varphi}_1)$, $\overline{M_2} := M_{\mathbb{Q}}(\overline{\theta}_2, \overline{\varphi}_2)$ as well as $\overline{M_1}^\theta := M^\theta_{\mathbb{Q}}(\overline{\theta}_1, \overline{\varphi}_1)$, $\overline{M_1}^\phi := M^\phi_{\mathbb{Q}}(\overline{\theta}_1, \overline{\varphi}_1)$ and analogously for $\overline{M_2}^\theta, \overline{M_2}^\phi$. Finally set

$$\begin{aligned} G^{\mathbb{Q}} := \langle R_{\mathbb{Q}}(\overline{\alpha})\overline{M_1}\widetilde{S}, w\rangle - \epsilon \cdot \big(|\langle R'_{\mathbb{Q}}(\overline{\alpha})\overline{M_1}\widetilde{S}, w\rangle| + |\langle R_{\mathbb{Q}}(\overline{\alpha})\overline{M_1}^\theta\widetilde{S}, w\rangle| + |\langle R_{\mathbb{Q}}(\overline{\alpha})\overline{M_1}^\phi\widetilde{S}, w\rangle|\big) \\ - 9\epsilon^2/2 - 4\kappa(1 + 3\epsilon), \end{aligned}$$

$$H^{\mathbb{Q}}_P := \langle \overline{M_2}P, w\rangle + \epsilon \cdot \big(|\langle \overline{M_2}^\theta P, w\rangle| + |\langle \overline{M_2}^\varphi P, w\rangle|\big) + 2\epsilon^2 + 3\kappa(1 + 2\epsilon).$$

If $G^{\mathbb{Q}} > \max_{P \in \widetilde{\mathbf{P}}} H^{\mathbb{Q}}_P$ then there does not exist a solution to Rupert's condition to $\mathbf{P}$ with

$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\overline{\theta}_1 \pm \epsilon, \overline{\varphi}_1 \pm \epsilon, \overline{\theta}_2 \pm \epsilon, \overline{\varphi}_2 \pm \epsilon, \overline{\alpha} \pm \epsilon].$$

Proof for Theorem 7.9

Definition 7.10

Let $\theta, \phi \in \mathbb{Q} \cap [-4, 4]$ and $M_{\mathbb{Q}} := M_{\mathbb{Q}}(\theta, \phi)$. Three points $\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3 \in \mathbb{Q}^3$ with $\|\widetilde{P}_1\|, \|\widetilde{P}_2\|, \|\widetilde{P}_3\| \leq 1 + \kappa$ are called ϵ-κ-spanning for (θ, ϕ) if it holds that:

$$\langle R(\pi/2) M_{\mathbb{Q}} \widetilde{P}_1, M_{\mathbb{Q}} \widetilde{P}_2 \rangle > 2\epsilon(\sqrt{2} + \epsilon) + 6\kappa,$$
$$\langle R(\pi/2) M_{\mathbb{Q}} \widetilde{P}_2, M_{\mathbb{Q}} \widetilde{P}_3 \rangle > 2\epsilon(\sqrt{2} + \epsilon) + 6\kappa,$$
$$\langle R(\pi/2) M_{\mathbb{Q}} \widetilde{P}_3, M_{\mathbb{Q}} \widetilde{P}_1 \rangle > 2\epsilon(\sqrt{2} + \epsilon) + 6\kappa.$$

Lemma 7.11

Let $P_1, P_2, P_3 \in \mathbb{R}^3$ with $\|P_i\| \leq 1$ and $\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3 \in \mathbb{Q}^3$ be their κ-rational approximations. Assume that $\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3$ are ϵ-κ-spanning for some $\theta, \phi \in \mathbb{Q} \cap [-4, 4]$, then P_1, P_2, P_3 are ϵ-spanning for θ, ϕ.

Proof for Lemma 7.11

See polyhedron.without.rupert, Lemma 46.

Lemma 7.12

Let $P, Q \in \mathbb{R}^3$ with $\|P\|, \|Q\| \leq 1$, and let $\widetilde{P}, \widetilde{Q}$ be κ-approximations. Then, for parameters in $[-4, 4]$,

- $|\langle X, P \rangle - \langle X_{\mathbb{Q}}, \widetilde{P} \rangle| \leq 3\kappa$
- $|\langle MP, MQ \rangle - \langle M_{\mathbb{Q}} \widetilde{P}, M_{\mathbb{Q}} \widetilde{Q} \rangle| \leq 5\kappa$
- $|\|MQ\| - \|M_{\mathbb{Q}} \widetilde{Q}\|| \leq 3\kappa$

Proof for Lemma 7.12

See polyhedron.without.rupert, Lemma 49.

Corollary 7.13

Let $P, Q \in \mathbb{R}^3$ with $\|P\|, \|Q\| \leq 1$ and $\widetilde{Q}$ a κ-rational approximation of Q. Let $\alpha, \theta, \phi, \overline{\theta}, \overline{\varphi} \in [-4, 4]$ and set $M = M(\theta, \phi)$, $M_{\mathbb{Q}} = M_{\mathbb{Q}}(\theta, \phi)$, $\overline{M} = M(\overline{\theta}, \overline{\varphi})$, $\overline{M}_{\mathbb{Q}} = M_{\mathbb{Q}}(\overline{\theta}, \overline{\varphi})$. Then

$$\big| \|R(\alpha) MP - \overline{M} Q\| - \|R_{\mathbb{Q}}(\alpha) M_{\mathbb{Q}} P - \overline{M}_{\mathbb{Q}} \widetilde{Q}\| \big| \leq 6\kappa$$

Note that the rational side uses P directly (not a rational approximation $\widetilde{P}$).

Proof for Corollary 7.13

See polyhedron.without.rupert, Corollary 50.

Corollary 7.14

In the setting of Lemma 7.12, let $\sqrt[+]{x}$ be an upper square-root function, i.e., $\sqrt{x} \leq \sqrt[+]{x}$ for all real $x \geq 0$ with rational output on rational input. Set $\|x\|_+ \coloneqq \sqrt[+]{\|x\|^2}$. Set

$$A = \frac{\langle MP, M(P-Q)\rangle - 2\epsilon\|P-Q\| \cdot (\sqrt{2}+\varepsilon)}{\left(\|MP\| + \sqrt{2}\varepsilon\right) \cdot \left(\|M(P-Q)\| + 2\sqrt{2}\varepsilon\right)}$$

as well as

$$A_{\mathbb{Q}} = \frac{\langle M_{\mathbb{Q}}\widetilde{P}, M_{\mathbb{Q}}(\widetilde{P}-\widetilde{Q})\rangle - 10\kappa - 2\epsilon(\|\widetilde{P}-\widetilde{Q}\| + 2\kappa) \cdot (\sqrt{2}+\varepsilon)}{\left(\|M_{\mathbb{Q}}\widetilde{P}\|_+ + \sqrt{2}\varepsilon + 3\kappa\right) \cdot \left(\|M_{\mathbb{Q}}(\widetilde{P}-\widetilde{Q})\|_+ + 2\sqrt{2}\varepsilon + 6\kappa\right)}.$$

Assume that $A \geq 0$. Then $A \geq A_{\mathbb{Q}}$.

Proof for Corollary 7.14

See polyhedron.without.rupert, Corollary 51.

Theorem 7.15

Let **P** be a polyhedron with radius $\rho = 1$ and $\widetilde{P}_i$ be a κ-rational approximation of $P_i \in$ **P**. Set $\widetilde{\mathbf{P}} = \{\widetilde{P}_i \text{ for } P_i \in \mathbf{P}\}$. Let $P_1, P_2, P_3, Q_1, Q_2, Q_3 \in$ **P** be not necessarily distinct and assume that P_1, P_2, P_3 and Q_1, Q_2, Q_3 are congruent. Let $\epsilon > 0$ and $\overline{\theta}_1, \overline{\varphi}_1, \overline{\theta}_2, \overline{\varphi}_2, \overline{\alpha} \in \mathbb{Q} \cap [-4, 4]$. Set $\overline{X_1} \coloneqq X_{\mathbb{Q}}(\overline{\theta}_1, \overline{\varphi}_1), \overline{X_2} \coloneqq X_{\mathbb{Q}}(\overline{\theta}_2, \overline{\varphi}_2)$ as well as $\overline{M_1} \coloneqq M_{\mathbb{Q}}(\overline{\theta}_1, \overline{\varphi}_1), \overline{M_2} \coloneqq M_{\mathbb{Q}}(\overline{\theta}_2, \overline{\varphi}_2)$. Assume that there exist $\sigma_P, \sigma_Q \in \{0, 1\}$ such that

$$(-1)^{\sigma_P}\langle \overline{X_1}, \widetilde{P}_i\rangle > \sqrt{2}\varepsilon + 3\kappa \quad \text{and} \quad (-1)^{\sigma_Q}\langle \overline{X_2}, \widetilde{Q}_i\rangle > \sqrt{2}\varepsilon + 3\kappa,$$

for all $i = 1, 2, 3$. Moreover, assume that $\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3$ are ϵ-κ-spanning for $(\overline{\theta}_1, \overline{\varphi}_1)$ and that $\widetilde{Q}_1, \widetilde{Q}_2, \widetilde{Q}_3$ are ϵ-κ-spanning for $(\overline{\theta}_2, \overline{\varphi}_2)$. Let $\sqrt[+]{x}$ and $\sqrt[-]{x}$ be upper- and lower-square-root functions, then set $\|Z\|_+ \coloneqq \sqrt[+]{\|Z\|^2}$ and $\|Z\|_- \coloneqq \sqrt[-]{\|Z\|^2}$ for $Z \in \mathbb{R}^n$. Finally, assume that for all $i = 1, 2, 3$ and any $\widetilde{Q}_j \in \widetilde{\mathbf{P}} \setminus \widetilde{Q}_i$ it holds that the rational inequality ($B_\epsilon^{\mathbb{Q}}$) from the paper is satisfied, for some $r > 0$ such that $\min_{i=1,2,3} \|\overline{M_2}\widetilde{Q}_i\|_- > r + \sqrt{2}\epsilon + 3\kappa$ and for some $\delta \in \mathbb{R}$ with

$$\delta \geq \max_{i=1,2,3} \left\| R_{\mathbb{Q}}(\overline{\alpha})\overline{M_1}\widetilde{P}_i - \overline{M_2}\widetilde{Q}_i \right\|_+ /2 + 3\kappa.$$

Then there exists no solution to Rupert's problem $R(\alpha)M(\theta_1, \phi_1)\mathbf{P} \subset M(\theta_2, \phi_2)\mathbf{P}^\circ$ with

$$(\theta_1, \phi_1, \theta_2, \phi_2, \alpha) \in [\overline{\theta}_1 \pm \epsilon, \overline{\varphi}_1 \pm \epsilon, \overline{\theta}_2 \pm \epsilon, \overline{\varphi}_2 \pm \epsilon, \overline{\alpha} \pm \epsilon] \subseteq \mathbb{R}^5.$$

Proof for Theorem 7.15

8 Computational Step

Theorem 8.1

There exists a valid solution table whose zeroth row covers

$$\begin{aligned}\theta_1, \theta_2 &\in [0, 2\pi/15] \subset [0, 0.42],\\ \varphi_1 &\in [0, \pi] \subset [0, 3.15],\\ \varphi_2 &\in [0, \pi/2] \subset [0, 1.58],\\ \alpha &\in [-\pi/2, \pi/2] \subset [-1.58, 1.58].\end{aligned}$$

Proof for Theorem 8.1

By exhibiting the table and running the validity checking algorithm.

Theorem 8.2

If a global node in the solution tree is valid, then there is no Rupert solution for its interval.

Proof for Theorem 8.2

Theorem 8.3

If a local node in the solution tree is valid, then there is no Rupert solution for its interval.

Proof for Theorem 8.3

Theorem 8.4

If we have a valid solution table, and in particular its ith row is valid, then there is no Rupert solution of the interval of its ith row.

Proof for Theorem 8.4

By a mutual induction on the number of rows left in the table following the ith. This is because validity constrains each row to only refer to later entries. Appeal inductively to this same theorem if the row splits into other nodes in the tree, or appeal to Theorem or Theorem at the leaves.

Corollary 8.5

If we have a valid solution table, then there is no Rupert solution of the interval of its zeroth row.

Proof for Corollary 8.5

Immediate special case of Theorem 8.4.

9 Main Theorems

Theorem 9.1

There does not in fact exist a noperthedron Rupert solution with

$$\begin{aligned}
\theta_1, \theta_2 &\in [0, 2\pi/15] \subset [0, 0.42],\\
\varphi_1 &\in [0, \pi] \subset [0, 3.15],\\
\varphi_2 &\in [0, \pi/2] \subset [0, 1.58],\\
\alpha &\in [-\pi/2, \pi/2] \subset [-1.58, 1.58].
\end{aligned}$$

Proof for Theorem 9.1

By this theorem, there is a valid solution table containing a valid row whose pose interval is a superset of the 5-dimensional interval above. By this theorem, there is no Rupert solution in that interval.

Theorem 9.2

There is no 5-parameter pose that makes the noperthedron have the Rupert property.

Proof for Theorem 9.2

Theorem Theorem 9.1 says there is no tight pose that makes the noperthedron Rupert. Corollary Corollary 2.2.2 says that this suffices for the general case.

Theorem 9.3

There is no purely rotational pose that makes the noperthedron have the Rupert property.

Proof for Theorem 9.3

Suppose there were a purely rotational pose. Then convert that to an equivalent 5-parameter pose with Theorem Theorem 4.2.1 and appeal to Theorem Theorem 9.2.

Theorem 9.4

> There is no pose that makes the noperthedron have the Rupert property.

Proof for Theorem 9.4

> By Theorem Theorem 4.3.1, we need only show that the noperthedron is pointsymmetric to see that if it is Rupert, then it must be Rupert via a purely rotational pose. But Lemma Lemma 2.1.9 shows exactly this. And yet we know via Theorem Theorem 9.3 that the noperthedron is not rotationally Rupert, so we have a contradiction, hence the noperthedron has no pose that makes it Rupert.

Theorem 9.5

> The noperthedron is not a Rupert set.

Proof for Theorem 9.5

> By Theorem Theorem 9.4, there is no pose that makes the noperthedron a Rupert set.

Theorem 9.6

> The noperthedron is not a Rupert polyhedron.

Proof for Theorem 9.6

> By Theorem 4.1.1, it suffices to show that the convex hull of noperthedron vertices is not a Rupert set, which is exactly the previous theorem.

Dependency Graph

The dependency graph is available in the HTML output.

Blueprint Summary

The Blueprint summary is available in the HTML output.

Blueprint Bibliography

polyhedron.without.rupert Jakob Steininger and Sergey Yurkevich, 2025. “A convex polyhedron without Rupert’s property”[1]. arXiv:polyhedron.without.rupert

[1] `https://arxiv.org/abs/polyhedron.without.rupert`